IMU Noise and Characterization

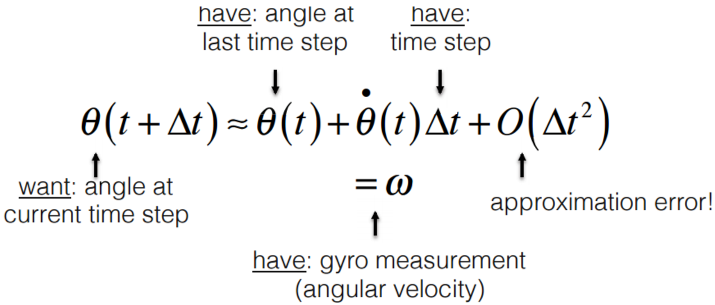
June 20, 2017

Outline

1. Motivation
2. Power Spectral Density
3. Allan Variance
4. IMU Noise Model
5. IMU Pre-Integration

1. Motivation

Motivation for Modelling IMU Noise



$$\theta(t+\Delta t) \approx \theta(t) + \dot{\theta}(t)\Delta t + O(\Delta t^2)$$

$$\dot{\theta}(t) = \omega$$

Figure: From Gyro Measurements to Orientation

Motivation for Modelling IMU Noise: Example

Gyro Integration: nonlinear motion, noise, bias

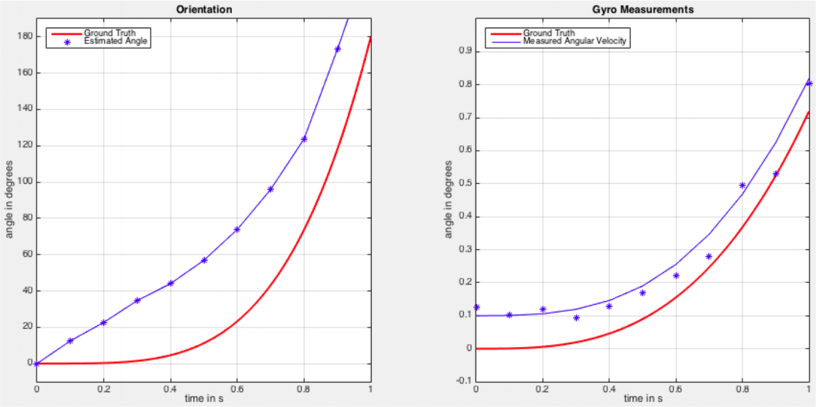


Figure: Error from integrating Gyro Measurements without dealing with noise

Motivation for Modelling IMU Noise: IMU Noise Components

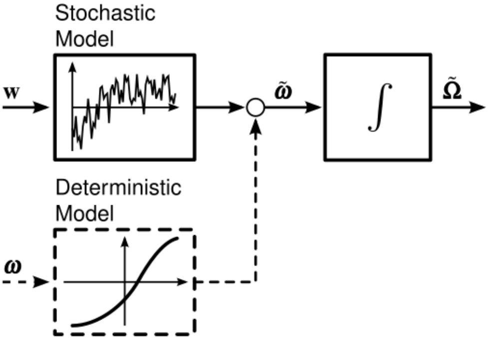


Fig. 1. Inertial sensor model with a deterministic and a random component. Here, the true angular rates $\boldsymbol{\omega}$ are corrupted with deterministic errors, for example a scale factor that varies with temperature, as well as non-deterministic errors, such as additive broadband noise. This report presents a method to identify noise processes according to their contribution to the angular increments $\tilde{\boldsymbol{\Omega}}$.

Figure: IMU Noise Components

Motivation for Modelling IMU Noise: Types of IMU Noise

- **Quantization Noise**
- **Angle / Velocity Random Walk Noise**
- **Correlated Noise**
- **Bias Instability Noise**
- **Rate / Acceleration Random Walk Noise**

2. Power Spectral Density

Fourier Transform

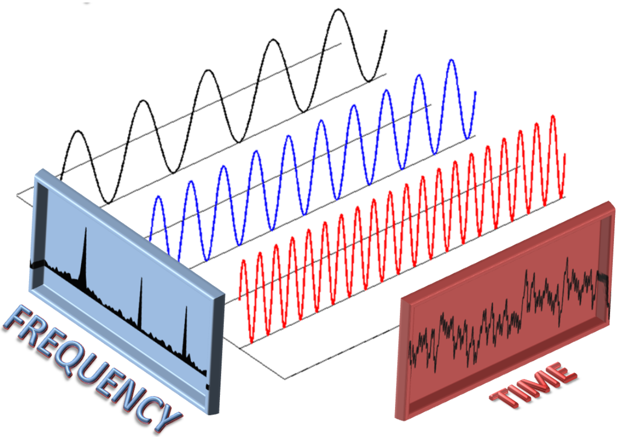


Figure: Fourier Transform

Power Spectral Density (PSD): Naming

- **Power**: refers to the fact that PSD is the **mean-square value** of the signal being analyzed
- **Spectral**: refers to the fact PSD is a function of frequency, it **represents distribution of a signal over a spectrum of frequencies**
- **Density**: refers to the fact that the **magnitude of PSD is normalized to a single hertz bandwidth.**

Power Spectral Density (PSD): Form

If the signal being analyzed is a **Wide-Sense Stationarity** (WSS) discrete **random process**, according to the **Wiener-Khinchin theorem** the PSD is defined as:

$$P(f) = \sum_{m=-\infty}^{\infty} R_{xx}(m) \exp(-j2\pi f m) \qquad (1)$$

Where $R_{xx}(f)$ is the **Autocorrelation function** of the random process $X(t)$ and τ is the time lag:

$$R_{xx}(f) = E[X(t)X(t-\tau)] \qquad (2)$$

Wide Sense Stationarity (WSS)

- A **Random Process is Stationary** if its statistical properties **do not change in time**
- WSS is also known as **Weak-Sense Stationarity**, **Covariance Stationarity** or **Second-Order Stationarity**.
- The main thing to know is a random process is WSS if its **mean** and its **correlation function** do not change by shifts in time.

Autocorrelation Function

Autocorrelation is the **degree of similarity between a given time series and a lagged version** of itself over successive time intervals

$$R_{xx}(f) = E[X(t)X(t-\tau)] \qquad (3)$$

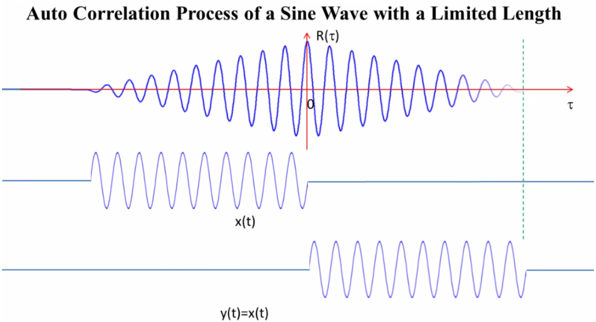


Figure: Autocorrelation in Action

Relationship between PSD and FT

- In **most practical situations**, the PSD of a random process is **not available.**
- Can **estimate** a given signal's power spectral density **by taking magnitude squared of its Fourier transform** as the estimate of the PSD

One form is:

$$P(f_k) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x_n \exp\left(-\frac{j2\pi kn}{N} \right) \right|^2 \tag{4}$$

If we compare **DFT**

$$|X(f_k)| = \left| \sum_{n=0}^{N-1} x[n] \exp\left(-j2\pi nk/N\right) \right| \tag{5}$$

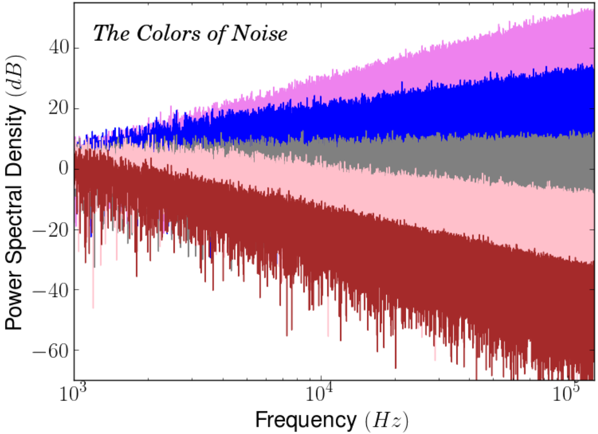
And **PSD (estimate)**

$$P(f_k) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x[n] \exp\left(-j2\pi f_k n\right) \right|^2 \tag{6}$$

You have:

$$P(f_k) = \frac{1}{N} |X(f_k)|^2 \tag{7}$$

Power Spectral Density: Color of Noise



Power Spectral Density

In summary:

- **DFT $\neq$ PSD**
- **DFT**: shows the spectral content of the signal (amplitude and phase of harmonics)
- **PSD**: describes how the power of the signal is distributed over frequency by performing the **mean-square** on the signal value.

3. Allan Variance

Allan Variance

Also called a **Two-Sample Deviation**, or square-root of the **Allan Variance**, where:

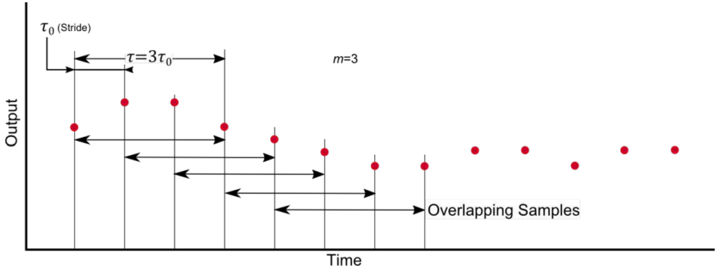
$$\sigma_y^2(\tau) = \frac{1}{2}\langle(\bar{y}_{n+1} - \bar{y}_n)^2\rangle \tag{8}$$

$$= \frac{1}{2\tau^2}\langle(y_{n+2} - 2y_{n+1} + y_n)^2\rangle \tag{9}$$

Where τ is the observation period, $\bar{y}_n$ is the n-th fractional frequency average over the observation time τ. The samples are taken with no dead-time between them, which is achieved by letting time period $T = \tau$.

Characterize IMU Noise with Allan Variance

1. Acquire time series data on gyroscope or accelerometer
2. Set average time to be $\tau = m\tau_0$, where m is the averaging factor. The value of m where $m < (N-1)/2$.
3. Divide time history of signal into clusters of finite time duration of $\tau = m\tau_0$



Characterize IMU Noise with Allan Variance

4. Once clusters are form, compute the Allan Variance
   - Calculate θ corresponding to each gyro output sample, this can be accomplished as in.

$$\theta(t) = \int^{t} \Omega(t')dt' \tag{10}$$

   - Once N values of θ have been computed, calculate the Allan Variance σ^2 represents as a function of τ where $\langle \cdot \rangle$ is the ensemble average.

$$\sigma^2 = \frac{1}{2\tau^2} \left\langle (\theta_{k+2m} - 2\theta_{k+m} + \theta_k)^2 \right\rangle \tag{11}$$

Characterize IMU Noise with Allan Variance

5. Calculate **Allan Deviation (AD)** value for a particular τ. This can be obtained simply by square rooting the Allan Variance (AVAR). This result will now be used to characterize the noise in a gyroscope.

$$\text{AD}(\tau) = \sqrt{\text{AVAR}(\tau)} \tag{12}$$

Characterize IMU Noise with Allan Variance

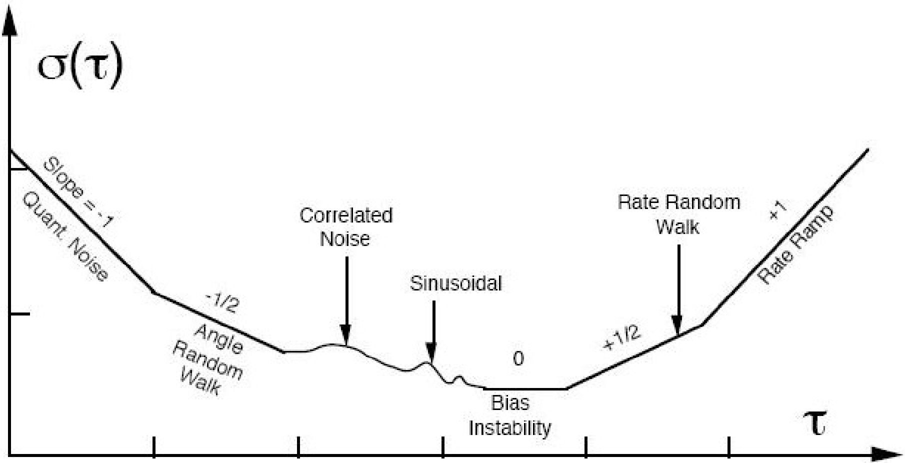


Figure: Characteristics of an Allan Deviation Plot (For Gyroscope)

Characterize IMU Noise with Allan Variance

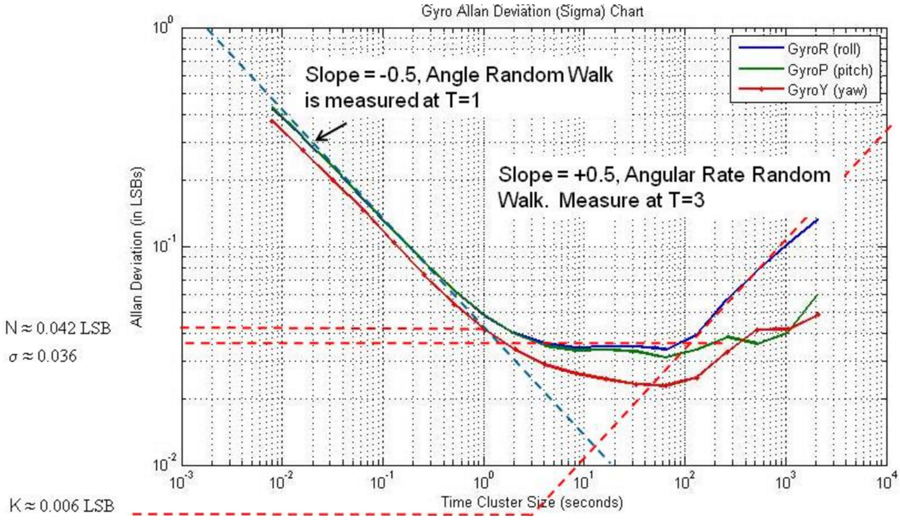


Figure: Gyrometer Noise Characterization

Characterize IMU Noise with Allan Variance

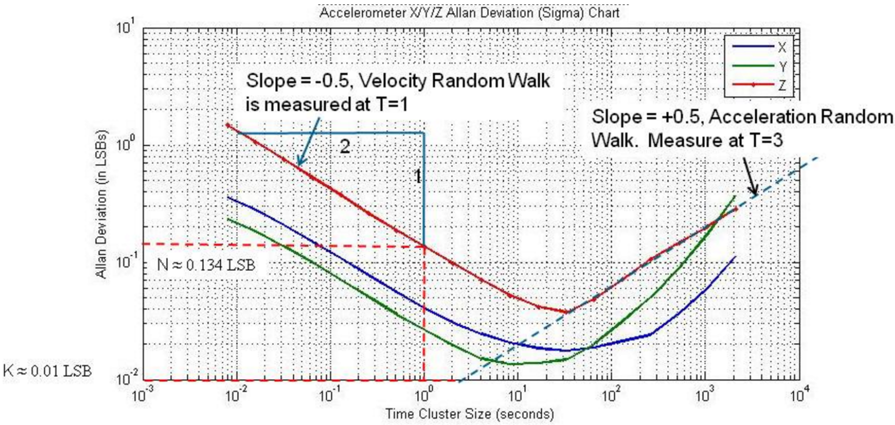


Figure: Accelerometer Noise Characterization

4. IMU Noise Model

IMU Noise Model

The standard noise model:

$$z = x + v \tag{13}$$

$$\dot{x} = \frac{1}{\tau_b} x + \omega \tag{14}$$

Where:

- z is the modelled noise process
- x is the slowly varying process with correlation time τ_b, "driven" by another independent white noise w.
- v is the white noise component

IMU Noise Model: Discrete version

Algorithm 1 Discrete-Time Equivalent of the Standard Noise Model (1a)

init:

$\sigma_{wd}^2 \leftarrow \frac{1}{\Delta t}\sigma_w^2$ ▷ assuming υ is band-limited to $\frac{1}{2\Delta t}$

$\sigma_{bd}^2 \leftarrow \Delta t \sigma_b^2$ ▷ assuming $\tau_b \gg \Delta t$

$$\Phi_d \leftarrow \exp\left(-\frac{1}{\tau_b}\Delta t\right)$$

$$x_0 \leftarrow \begin{cases} 0 & \text{if } \frac{1}{\tau_b} = 0 \text{ (by definition)} \\ \mathcal{N}\left(0, \frac{\sigma_b^2 \tau_b}{2}\right) & \text{otherwise} \end{cases}$$

for $k \leftarrow 1$ **to** n **do**

$w_k \leftarrow \mathcal{N}\left(0, \sigma_{bd}^2\right)$, $v_k \leftarrow \mathcal{N}\left(0, \sigma_{wd}^2\right)$

$x_k \leftarrow \Phi_d x_{k-1} + w_k$

$z_k \leftarrow x_k + v_k$

end for

Figure: IMU Noise Model: Discrete version

5. IMU Pre-Integration

IMU Pre-Integration

Realtime is difficult as map and trajectory grows overtime, there are generally 3 approaches towards realtime operation:

- PTAM
- Marginalization (fixed-lag smoothing)
- Filtering

But PTAM has a keyframe limit, filtering and marginalization commit to a linearization point when marginalizing which introduces drift and potential inconsistencies.

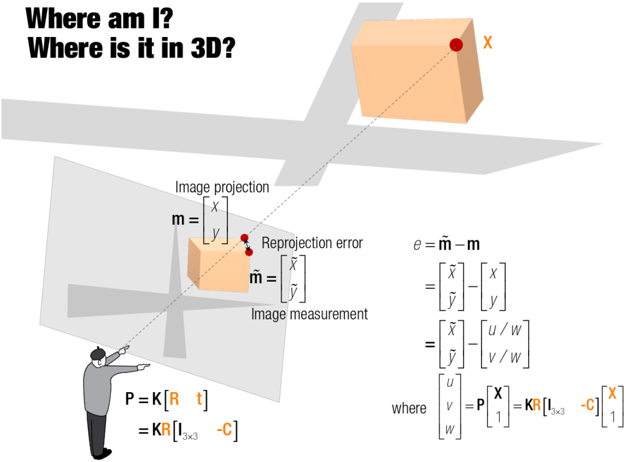
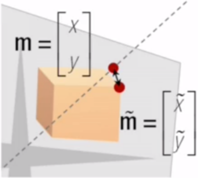


Figure: Bundle Adjustment

IMU Pre-Integration: Bundle Adjustment (structured)

Reprojection error

$$e = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} - \begin{bmatrix} u/w \\ v/w \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{R}\left[\mathbf{I}_{3\times 3} \quad -\mathbf{C} \right] \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$



$$\underset{\mathbf{R},\mathbf{C},\mathbf{X}}{\text{minimize}} \left\| \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} - \begin{bmatrix} u(\mathbf{R},\mathbf{C},\mathbf{X}) / w(\mathbf{R},\mathbf{C},\mathbf{X}) \\ v(\mathbf{R},\mathbf{C},\mathbf{X}) / w(\mathbf{R},\mathbf{C},\mathbf{X}) \end{bmatrix} \right\|^2$$

$$= \underset{\mathbf{q},\mathbf{C},\mathbf{X}}{\text{minimize}} \left\| \underbrace{\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}}_{\mathbf{b}} - \underbrace{\begin{bmatrix} u(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) / w(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) \\ v(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) / w(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) \end{bmatrix}}_{\mathbf{f}(\mathbf{R},\mathbf{C},\mathbf{X})} \right\|^2$$

: Quaternion parameterization

IMU Pre-Integration: Approach

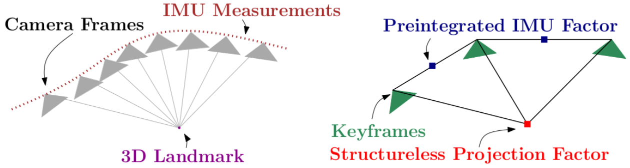


Fig. 4: Left: visual and inertial measurements in VIN. Right: factor graph in which several IMU measurements are summarized in a single preintegrated IMU factor and a structureless vision factor constraints keyframes observing the same landmark.

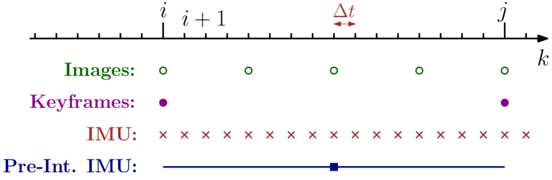


Fig. 5: Different rates for IMU and camera.

IMU Pre-Integration: Key things to note

- **Avoid repeated integration** by defining relative motion increments
- **Assume bias** is known and constant
- Make **Bundle Adjustment problem structureless** by "Lifting" the cost function

IMU Pre-Integration: Results

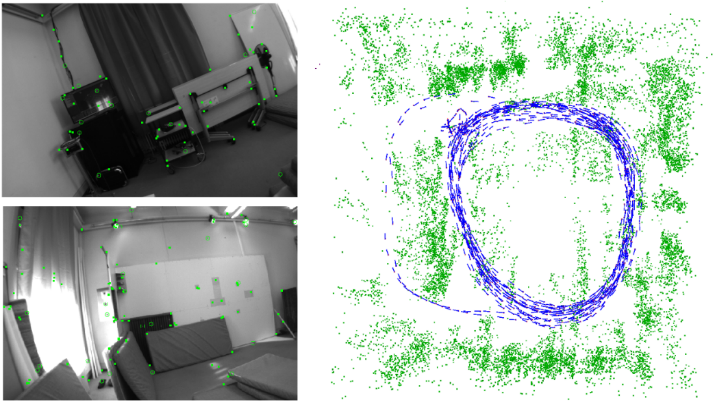


Fig. 6: Left: two images from the indoor trajectory dataset with tracked features in green. Right: top view of the trajectory estimate produced by our approach (blue) and 3D landmarks triangulated from the trajectory (green).

Figure: IMU Pre-integration Results

IMU Pre-Integration: Results

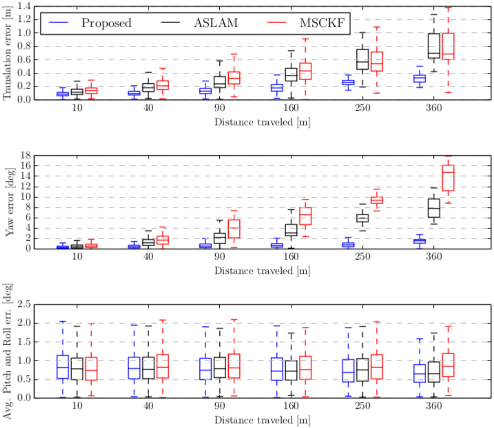


Fig. 7: Comparison of the proposed approach versus the ASLAM algorithm [9] and an implementation of the MSCKF filter [20]. Relative errors are measured over different segments of the trajectory, of length $\{10, 40, 90, 160, 250, 360\}$m, according to the odometric error metric in [46].

Figure: IMU Pre-integration Results

Questions?

Questions?

Homework

1. What are the definitions of these terms?
   - **Quantization Noise**
   - **Angle / Velocity Random Walk Noise**
   - **Correlated Noise**
   - **Bias Instability Noise**
   - **Rate / Acceleration Random Walk Noise**
2. Simulate an IMU using the standard noise model
3. Plot Fourier Transform and Power Spectral Density of simulated IMU
4. Extract the IMU Noise characteristics using Allan Variance