Rotations and Transformations

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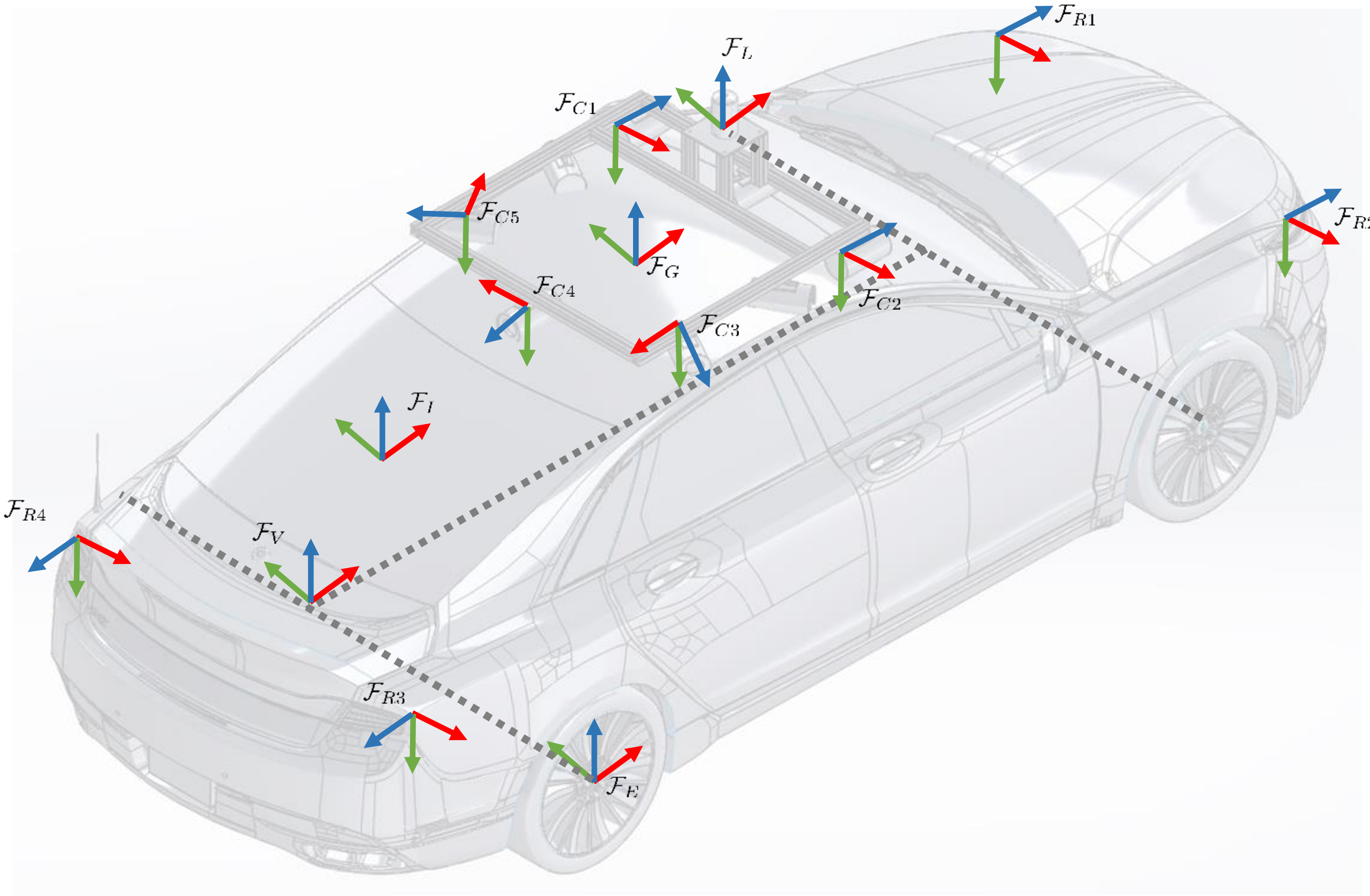


How do we map and analyze quantities between co-ordinate frames?
6 DOF, translation in space and rotation of coordinate axes

- Mathematical Preliminaries
  - Algebraic Structures
  - Topological Spaces and Manifolds
  - Matrix Lie Groups, SO(3) and SE(3)
- Rotation Representations
  - Euler Angles
  - Quaternions
  - Rotation Matrices
- Transformations

- Transformations from one coordinate frame to another can be described using standard geometric structures

- A brief intro to the definitions we will rely on, and what all the terms mean follows

- This field works from first principles to define the minimum properties needed to create well known types of structures

  - What are the minimum properties (axioms) to define integers, real numbers, etc.?
    - Groups, Rings, Fields
    - Vectors spaces

- Warning! Needs further expansion and clarification

A **group, $\boldsymbol{G}$,** is set of elements with an operation, $\circ : \boldsymbol{G} \times \boldsymbol{G} \rightarrow \boldsymbol{G}$, which satisfies the following four axioms for all $g \in G$:

Closed: $g \circ g \in G$

Associative: $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$

Identify element, e : $e \cdot g = g \cdot e = g$

Inverse element, g^{-1} : $g^{-1} \cdot g = e$

Examples:

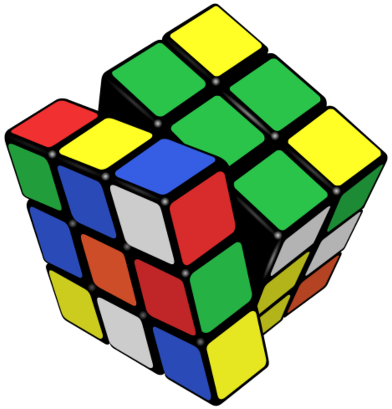
Integers under addition (not multiplication, no inverse)

$$a + b = c, \quad e = 0, \quad a^{-1} = -a$$

Fractions under multiplication

$$\frac{a}{b}\frac{c}{d} = \frac{ac}{bd} \qquad e = 1 \qquad \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

Rubic's cube patterns under rotations



- An example group is the set of all invertible, $n \times n$ –matrices called the general linear group, **GL(n)**

- With respect to matrix multiplication this group is closed and all axioms above hold
  - Check!

- GL(n) consists of $\boldsymbol{A} \in M(n)$ for which $\det(\boldsymbol{A}) \neq \mathbf{0}$

A **ring,** $\boldsymbol{R}$**,** is set of elements with two operations, +: $\boldsymbol{R} \times \boldsymbol{R} \rightarrow \boldsymbol{R}$, and ∘: $\boldsymbol{R} \times \boldsymbol{R} \rightarrow \boldsymbol{R}$ which satisfy the following axioms for all $r \in R$:

($\boldsymbol{R}$, +) is an Abelian Group (a group for which a+b=b+a)

($\boldsymbol{R}$, ∘) is associative: $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$

Multiplication is distributive w.r.t. addition

$$g_1 \circ (g_2 + g_3) = (g_1 \circ g_2) + (g_2 \circ g_3)$$

In fact, integers are actually a ring

(a group with these extra properties)

A **field,** $\boldsymbol{F}$**,** is set of elements with two operations, +: $\boldsymbol{F} \times \boldsymbol{F} \rightarrow \boldsymbol{F}$, and ∘: $\boldsymbol{F} \times \boldsymbol{F} \rightarrow \boldsymbol{F}$ which satisfy the following axioms for all $f \in \boldsymbol{F}$:

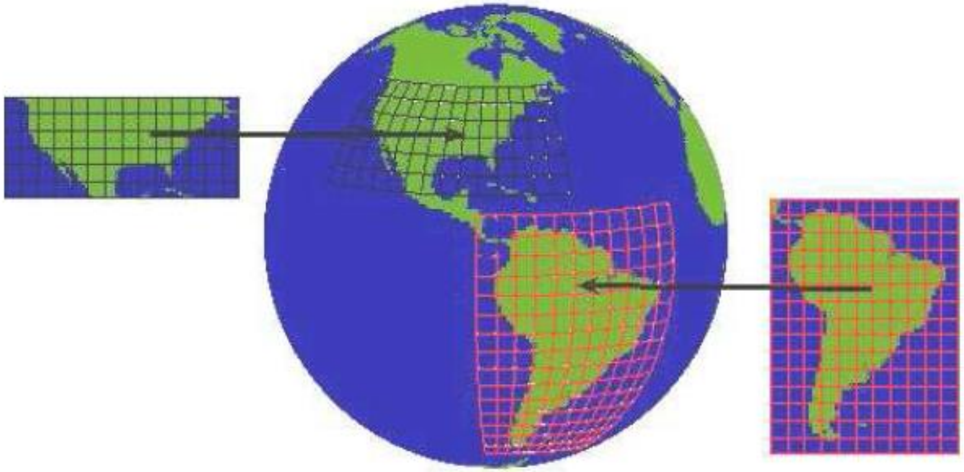
($\boldsymbol{F}$, +) is an Abelian Group (a group for which $a+b=b+a$)

($\boldsymbol{F}$, ∘) is an Abelian Group (a group for which $a \circ b=b \circ a$)

Rational numbers, real numbers and complex numbers are all fields

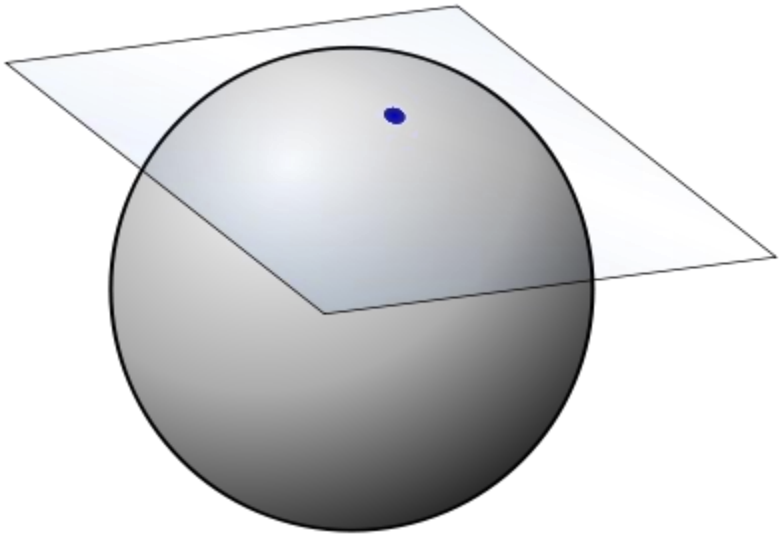
- A **vector space,** $\boldsymbol{V}$**,** over a field **,** $\boldsymbol{F}$**,** is set of elements of **,** $\boldsymbol{V}$**,** with two operations,addition, + : $\boldsymbol{V} \times \boldsymbol{V} \rightarrow \boldsymbol{V}$, and scalar multiplication ∘ **:** $\boldsymbol{F} \times \boldsymbol{V} \rightarrow \boldsymbol{V}$ which satisfy the following axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \boldsymbol{V}$, a,b $\in \boldsymbol{F}$:
  - Associativity of addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
  - Commutativity of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - Identity element of addition: $\mathbf{0} \in V$, $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
  - Inverse elements of addition: $\mathbf{v} \in$ V, $-\mathbf{v} \in V$, $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
  - Compatibility of scalar multiplication with field multiplication: $a(b\mathbf{v}) = (ab)\mathbf{v}$
  - Identity element of scalar multiplication: $\mathbf{I}\ \mathbf{v} = \mathbf{v}$
  - Distributivity of scalar multiplication w.r.t vector addition: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
  - Distributivity of scalar multiplication with respect to field addition: $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

- We are interested in Euclidean vector spaces, where every vector encodes a magnitude and direction,
  - Translations live in a vector space
  - Rotations live in a group



A manifold is a topological space that locally looks like an open subset of $\mathbb{R}^n$

Each point of an n-dimensional manifold has a neighbourhood that is homeomorphic to Euclidean space of dimension n.



Tangent Space: A vector space that best approximates the manifold about a point, tangent to the manifold at the point.

A **Lie Group** is a both a group and a smooth manifold such that the maps

$$G \mapsto G \qquad\qquad G \times G \mapsto G$$
$$g \mapsto g^{-1} \qquad\qquad (g, h) \mapsto g \cdot h$$

are smooth, (C^∞) , meaning the maps are differentiable so small deviations are continuous

Rotations SO(3) and Transformations SE(3) are examples of Lie Groups

- Rotations are part of a **special Lie group**
  - Special: det($\boldsymbol{R}$) =1
  - Orthogonal: matrix rows/columns are orthogonal

- SO(3) or Special Orthogonal Group

$$\mathbb{SO}(3) = \{R \in \mathbb{R}^{3\times 3} : R^\top R = I = RR^\top, \det(R) = 1\}$$

- An element of a Euclidean Group, E(n) is combines a translation and an orthogonal rotation A,

$$x \in E(n) \qquad \mathbb{R}^n \mapsto \mathbb{R}^n$$
$$A \in O(n) \qquad x \mapsto Ax + b$$
$$b \in \mathbb{R}^3$$

- When A is from SO(3), we get the **Special Euclidean Group**

$$\mathbb{SE}(3) = \left\{ T \in \mathbb{R}^{4\times 4}, \left[\begin{array}{c|c} R & t \\ \hline 0^{1\times 3} & 1 \end{array}\right] : R \in \mathbb{SO}(3), t \in \mathbb{R}^3 \right\}$$

- **Homeomorphic** to

$$\mathbb{SO}(3) \times \mathbb{R}^3$$

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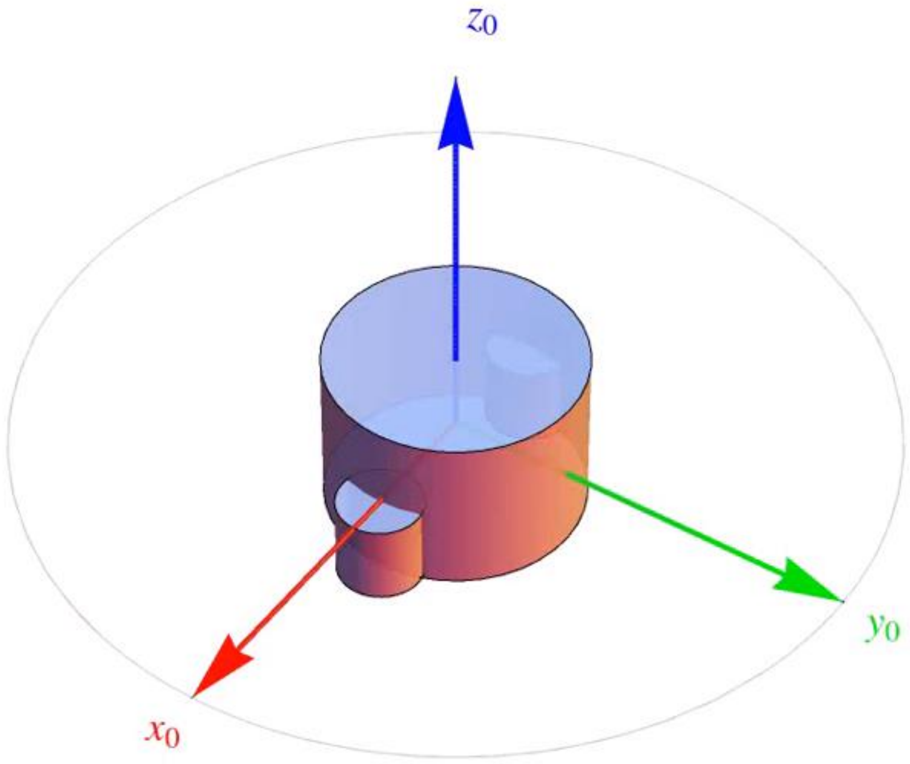
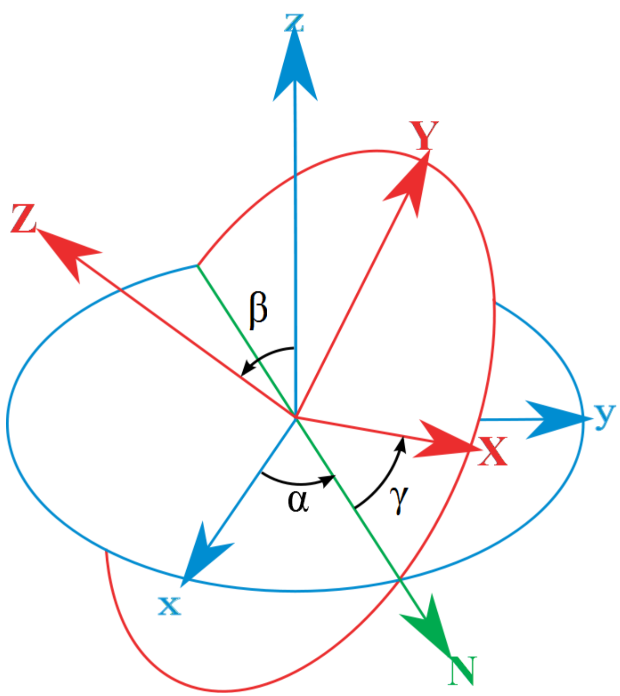


z_0
y_0
x_0

- Euler's Theorem: *Any two independent orthonormal coordinate frames can be related by a sequence of at most three rotations about coordinate axes, where no two successive rotations may be about the same axis*
- Given First Axes (xyz), rotate to Second Axes (XYZ) through 3 successive rotations
  - Rotation 1: About z by α
  - Rotation 2: About N by β
  - Rotation 3: About Z by γ
- Known as 3-1-3 Euler Angles



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- Aero convention: 3-2-1 Euler Angles ϕ, θ, ψ
  - Roll, Pitch, Yaw (when decoupled):
  - Rotation Matrices
    - 3 - Yaw
    - 2- Roll
    - 1- Pitch

$$R(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

- Direction Cosine Matrix (DCM)
  - All three rotations combined (from inertial to body)

$$R_I^B = R_{\phi,1} R_{\theta,2} R_{\psi,3}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

- Euler angles are measured relative to intermediate coordinate frames (3-2-1),
  - Not a rotation matrix

$$\omega_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

- Resulting transformations

R_e

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$\bar{R}_e$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \dfrac{\sin\phi}{\cos\theta} & \dfrac{\cos\phi}{\cos\theta} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$


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- An alternative way of representing rotations is through quaternions
  - Hamilton (1843) was looking for a field of dimension 4, to complete the picture (reals are a field of dimension 1, complex are a field of dimension 2)
    - Was only able to find a non-commutative division ring
    - He called them quaternions
    - While walking with his wife in Dublin, scribbled the rule of quaternions on a bridge so he would not forget it.

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

    - Everything but commutative multiplication work for quaternions (almost a field)

- Quaternions are a 4-tuple, divided into a scalar and a 3-vector
  - Let

$$\mathbf{i} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$
$$\mathbf{j} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$
$$\mathbf{k} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

  - Then a quaternion $q = (q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ can be written as

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

  - Addition simply adds the elements

$$q + p = (q_0 + p_0) + (q_1 + p_1)\mathbf{i} + (q_2 + p_2)\mathbf{j} + (q_3 + p_3)\mathbf{k}$$

- Quaternions are a 4-tuple, divided into a scalar and a 3-vector $cq = cq_0 + cq_1\mathbf{i} + cq_2\mathbf{j} + cq_3\mathbf{k}$
  - Multiplication by a constant

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

  - The product of two quaternions is defined by Hamilton's rule

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$$
$$\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$$
$$\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$$

- To get the rule for multiplication, do it out longhand and simplify

- Let $pq = (p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})$, then

$$p = (p_0, \mathbf{p}), q = (q_0, \mathbf{q})$$

$$r = pq = \underbrace{p_0 q_0 - \mathbf{p}\cdot\mathbf{q}}_{\text{Scalar part, } r_0} + \underbrace{p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}}_{\text{Vector part, } \mathbf{r}}$$

- In matrix form,

$$r = pq = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

- The complex conjugate of a quaternion is similar to complex numbers

$$q^* = q_0 - \mathbf{q}$$

  - Which leads to

$$q + q^* = 2q_0$$

  - And

$$(pq)^* = q^* p^*$$

- The 2-norm of a quaternion is $N(q) = \| q \|$

$$\| q \|^2 = qq^*$$
$$= q_0 q_0 - (-\mathbf{q}) \cdot \mathbf{q} + q_0 \mathbf{q} + q_0 (-\mathbf{q}) + (-\mathbf{q}) \times \mathbf{q}$$
$$= q_0^2 + \mathbf{q} \cdot \mathbf{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

- The inverse of a quaternion

$$q^{-1}q = qq^{-1} = 1$$

$$q^{-1}qq^{*} = q^{*}qq^{-1} = q^{*}$$

$$q^{-1} = \frac{q^{*}}{\|q\|^{2}}$$

- And if the quaternion is a unit quaternion

$$q^{-1} = q^{*}$$

  - Which is similar to a rotation matrix

- Unit quaternions can be related to an angle (and a vector), similar to the rotation matrix

$$q_0^2 + \|\mathbf{q}\|^2 = 1 \qquad \cos^2\theta + \sin^2\theta = 1$$

  - Therefore, there must exist an angle $\theta \in (-\pi, \pi]$ for any quaternion q
  - Then

$$\mathbf{u} = \frac{\mathbf{q}}{\|\mathbf{q}\|} = \frac{\mathbf{q}}{\sin\theta}$$

  - And we can express the unit quaternion and its conjugate as

$$q = \cos\theta + \mathbf{u}\sin\theta$$

$$q^* = \cos\theta - \mathbf{u}\sin\theta$$

- Define the unit quaternion rotation operator as

$$R_q(v) = qvq^*$$

  - Where *v* is the quaternion version of a vector **v** with zero scalar part (*v*=(0,**v**)) and quaternion multiplication is used. Simplifying yields

$$R_q(v) = (q_0^2 - \|\mathbf{q}\|^2)\mathbf{v} + 2(\mathbf{q}\cdot\mathbf{v}) + 2q_0(\mathbf{q}\times\mathbf{v})$$

  - The unit quaternion rotation operator is linear
    - Satisfies additivity and distributivity
  - The norm $||R_q(v)||$ is still $||\mathbf{v}||$
  - So it appears we might be on to something, here
    - If quaternions represent a rotation, then the rotation operation becomes linear in quaternion space

- Theorem: *The quaternion rotation operator $R_q(v)$ performs a rotation of* $\mathbf{v}$ *by* 2θ *about* $\mathbf{q}$.

- Proof: Define the components of $\mathbf{v}$ in the direction of and perpendicular to $\mathbf{q}$ ($\mathbf{a}$ and $\mathbf{n}$, respectively).

$$\mathbf{v} = \mathbf{a} + \mathbf{n}$$

- Which implies

$$\mathbf{a} = k\mathbf{q}$$

- By linearity and the definition of the rotation operator

$$R_q(a) = R_q(kq) = kR_q(q) = kq = a = (0, \mathbf{a})$$

- So the component of **v** along **q** is invariant to rotation, as required.
- For the perpendicular component, we must show that a rotation by 2θ occurs
- Expanding

$$R_q(n) = (q_0^2 - \|\mathbf{q}\|^2)\mathbf{n} + 2(\mathbf{q}\cdot\mathbf{n}) + 2q_0(\mathbf{q}\times\mathbf{n})$$
$$= (q_0^2 - \|\mathbf{q}\|^2)\mathbf{n} + 2q_0\|\mathbf{q}\|(\mathbf{u}\times\mathbf{n})$$

- Denote $\mathbf{n}_\perp = \mathbf{u}\times\mathbf{n}$

$$R_q(n) = (q_0^2 - \|\mathbf{q}\|^2)\mathbf{n} + 2q_0\|\mathbf{q}\|\mathbf{n}_\perp$$

- Note that

$$\| \mathbf{n}_{\perp} \| = \| \mathbf{u} \times \mathbf{n} \| = \| \mathbf{u} \| \| \mathbf{n} \| \sin \pi / 2 = \| \mathbf{n} \|$$

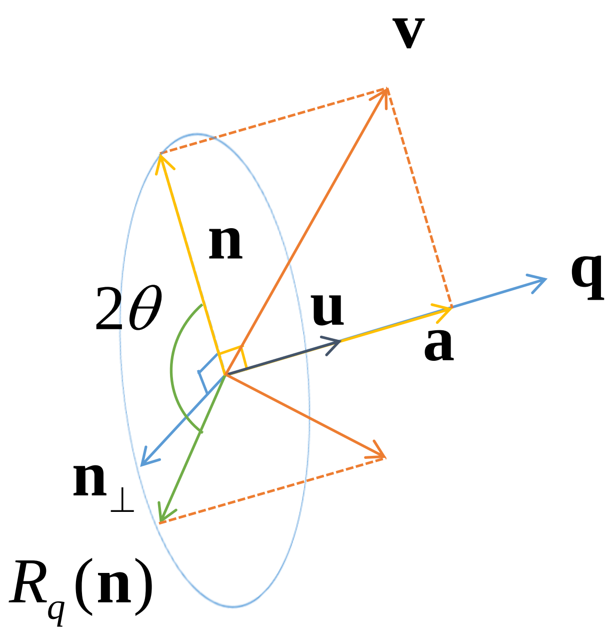
- Finally, using angle description of *q*

$$R_q(n) = (\cos^2\theta - \sin^2\theta)\mathbf{n} + 2\cos\theta\sin\theta\mathbf{n}_{\perp}$$
$$= \cos 2\theta\mathbf{n} + \sin 2\theta\mathbf{n}_{\perp}$$

  - But this is just a rotation of the component of **v** perpendicular to **q** in the plane by 2θ.

Quaternions for rotations

- The picture



- So now we have a physical interpretation of the quaternion as a combination of the Euler rotation vector **v=q** and angle $\gamma=2\theta$

- Going back to rotation operator, we can write it in matrix form and extract a conversion to rotation matrix

$$\mathbf{v}' = \begin{bmatrix} 2(q_0^2+q_1^2)-1 & 2q_1q_2-2q_0q_3 & 2q_1q_3+2q_0q_2 \\ 2q_1q_2+2q_0q_3 & 2(q_0^2+q_2^2)-1 & 2q_2q_3-2q_0q_1 \\ 2q_1q_3-2q_0q_2 & 2q_2q_3+2q_0q_1 & 2(q_0^2+q_3^2)-1 \end{bmatrix}\mathbf{v}$$

$$= R\mathbf{v}$$

- Similar to the rotation matrix and Euler angle update, quaternions can be updated directly from body rotation rates

- Body rotation rate quaternion (notation abuse)

$$\omega_B = (0, \boldsymbol{\omega}_B)$$

- Given a vector **v** with quaternion *v*= (0,**v**) and a unit quaternion *q* defining a rotation about **q** by 2θ

$$v' = qvq^{-1} \quad \Rightarrow \quad q^{-1}v' = vq^{-1}$$

$$v'q = qv$$

- Differentiating yields

$$\frac{dv'}{dt} = \frac{dq}{dt} v q^{-1} + q v \frac{dq^{-1}}{dt}$$

- Rearranging

$$\frac{dv'}{dt} = \frac{dq}{dt} q^{-1} v' + v' q \frac{dq^{-1}}{dt}$$

- From $qq^{-1} = 1$, we get

$$\frac{dq}{dt} q^{-1} + q \frac{dq^{-1}}{dt} = 0$$

- And combining leads to

$$\frac{dv'}{dt} = \frac{dq}{dt} q^{-1} v' - v' \frac{dq}{dt} q^{-1}$$

- Now define

$$p = \frac{dq}{dt} q^{-1}$$

- So we get

$$\frac{dv'}{dt} = pv' - v'p$$

- Recall that the scalar and vector parts of a quaternion multiplication are defined by

$$r = pq = \underbrace{p_0 q_0 - \mathbf{p}\cdot\mathbf{q}}_{\text{Scalar part, } r_0} + \underbrace{p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}}_{\text{Vector part, } \mathbf{r}}$$

- The scalar part of *v'* is 0 and, it turns out, the scalar part of *dv'/dt* is too

$$\frac{dv'}{dt} = pv' - v'p$$

$$\frac{dv_0^{'}}{dt} = p_0 v_0^{'} - \mathbf{p}\cdot\mathbf{v} - v_0^{'} p_0 + \mathbf{v}\cdot\mathbf{p}$$
$$= 0$$

- The vector part of *dv'/dt* returns

$$\frac{dv'}{dt} = pv' - v'p$$

$$\mathbf{r} = p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}$$

$$\frac{d\mathbf{v}'}{dt} = p_0\mathbf{v}' + v_0'\mathbf{p} + \mathbf{p}\times\mathbf{v} - v_0'\mathbf{p} - p_0\mathbf{v}' - \mathbf{v}\times\mathbf{p}$$

$$= 2(\mathbf{p}\times\mathbf{v}')$$

- So *d***v***'/dt* is a vector defined by a cross product
- We also know that *d***v***'/dt* is defined by the vector **v'** and its rate of rotation

$$\frac{d\mathbf{v}'}{dt} = \boldsymbol{\omega}_I \times \mathbf{v}' \quad \Longrightarrow \quad 2\mathbf{p} = \boldsymbol{\omega}_I$$

- Looking at *p*, we see that

$$\omega_B = 2p$$
$$= 2\frac{dq}{dt}q^{-1}$$

- And so we can update our quaternion as follows (with quaternion multiplication)

$$\dot{q} = \frac{1}{2}\omega_I q = \frac{1}{2}q\omega_B q^{-1}q = \frac{1}{2}q\omega_B$$

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As defined above, the Special Orthogonal group can also represent rotations

$$\mathbb{SO}(3) = \{R \in \mathbb{R}^{3\times 3} : R^\top R = I = RR^\top, \det(R) = 1\}$$

Not all 3x3 matrices are members of SO(3)

- Recall we have constraints on the rotation matrix.
- Locally, the group is 3 dimensional.
- 3 dimensional manifold embedded in

$$\mathbb{R}^{3\times 3}$$

- What matrices in SO(3) differ from the identity by a small amount?

A Rotation matrix near identity:

$$R = \begin{bmatrix} 1+a & b & c \\ d & 1+e & f \\ g & h & 1+i \end{bmatrix} \quad \text{where } a, b, c, d, e, f, g, h, i \text{ are small quantities}$$

The rotation times its transpose is identity:

$$\begin{bmatrix} 1+a & d & g \\ b & 1+e & h \\ c & f & 1+i \end{bmatrix} \begin{bmatrix} 1+a & b & c \\ d & 1+e & f \\ g & h & 1+i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+2a+a^2+d^2+g^2 & b+d+ab+de+gh & c+g+ac+df+gi \\ b+d+ab+de+gh & 1+2e+b^2+e^2+h^2 & f+h+bc+ef+hi \\ c+g+ac+df+gi & f+h+bc+ef+hi & 1+2i+c^2+f^2+i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ignoring second order terms:

$$\begin{bmatrix} 1+2a & b+d & c+g \\ b+d & 1+2e & f+h \\ c+g & f+h & 1+2i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Six independent constraints:

$$\begin{matrix} a=0 & e=0 & i=0 \\ b+d=0 & c+g=0 & f+h=0 \end{matrix}$$

Only need **three parameters** (b,c,f)

$$R = \begin{bmatrix} 1+a & b & c \\ d & 1+e & f \\ g & h & 1+i \end{bmatrix} \qquad R = \begin{bmatrix} 1 & b & c \\ -b & 1 & f \\ -c & -f & 1 \end{bmatrix}$$

Any rotation near the identity looks like

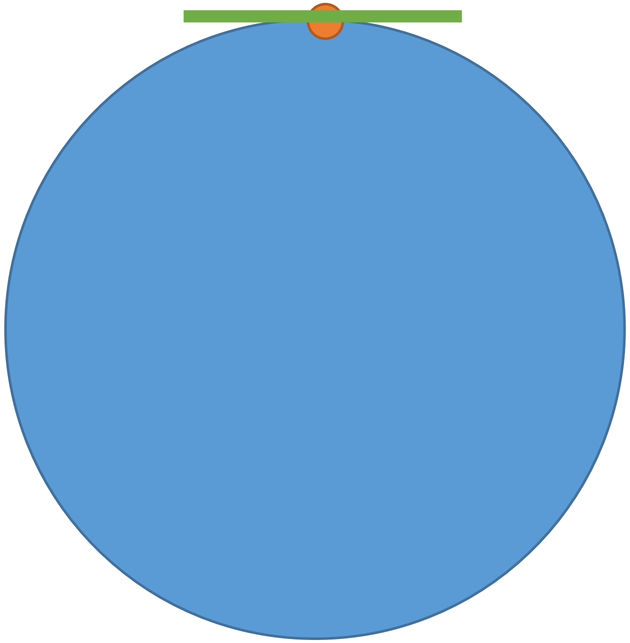
$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - b \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

G1,G2,G3 are called **Generators**

$$R = I + \alpha_1 G_1 + \alpha_2 G_2 + \alpha_3 G_3 \qquad \text{where } \alpha_1, \alpha_2, \alpha_3 \text{ are infinitessimal}$$

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad G_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

All group elements have a tangent space. Together, known as a **Tangent Bundle**



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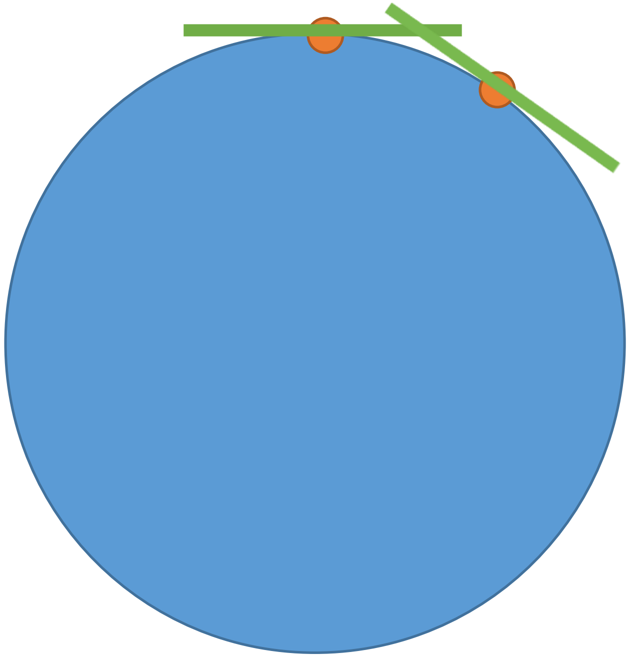

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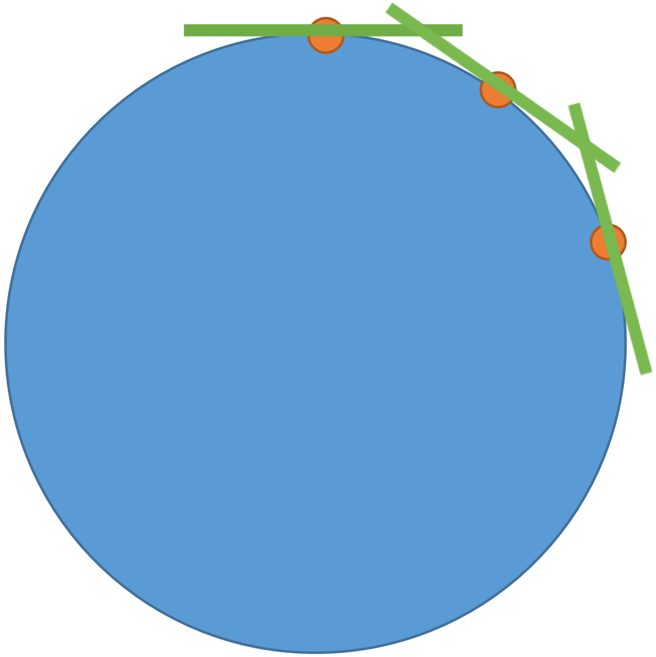
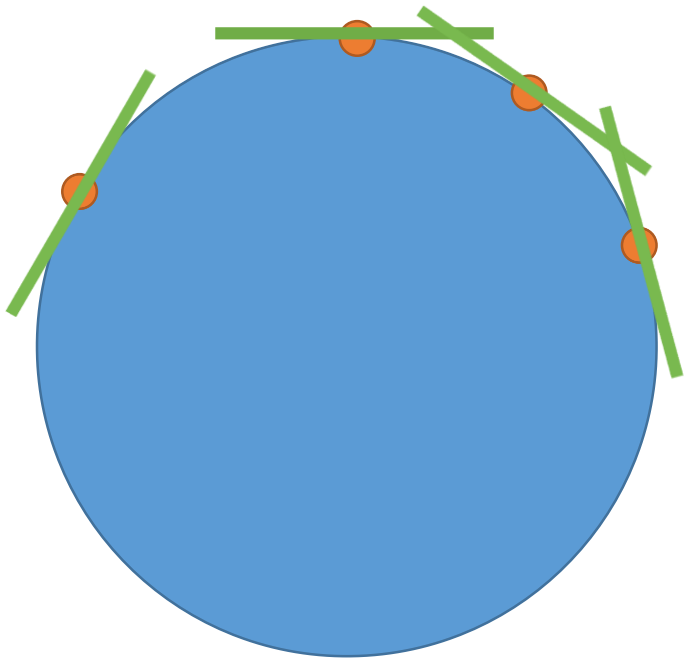


All group elements have a tangent space. Together, known as a **Tangent Bundle**



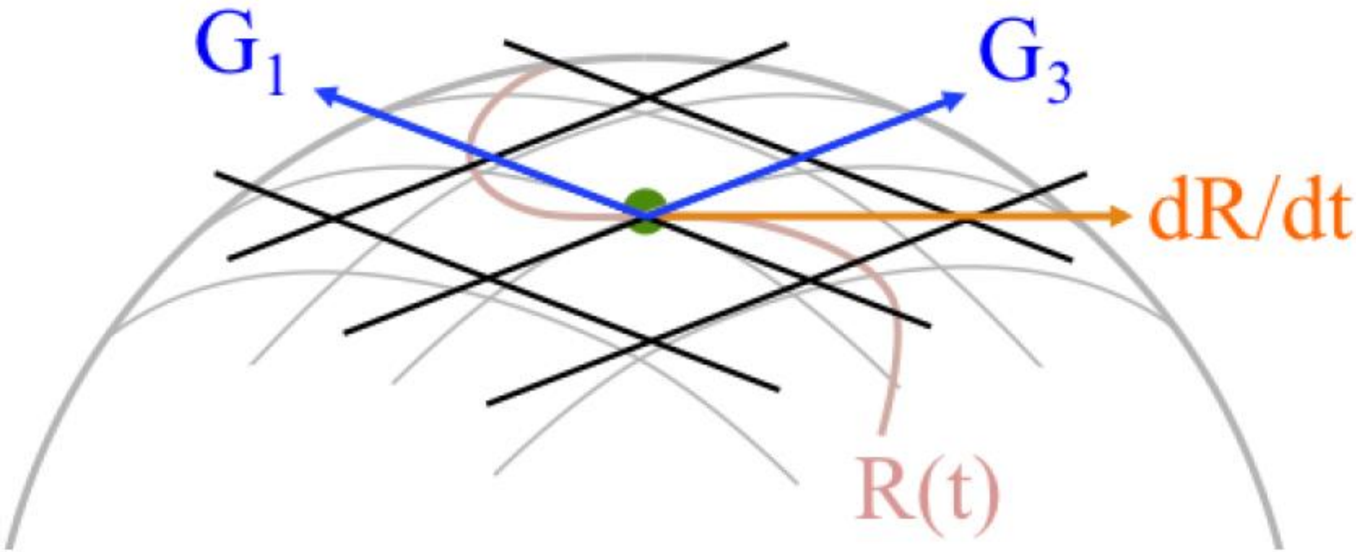
Let's re-visit our SO(3) example

$$R = I + \alpha_1 G_1 + \alpha_2 G_2 + \alpha_3 G_3 \qquad \text{where } \alpha_1, \alpha_2, \alpha_3 \text{ are infinitessimal}$$

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad G_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we let the generator coefficients become non infinitesimal, we get a 3-dimensional space that is tangent to the Identity element. Note that the generators form a **basis** for this space.

2D visualization



The tangent space at the identity element is given a special designation: the **Lie Algebra** of the group, and is denoted by $\mathfrak{g}$

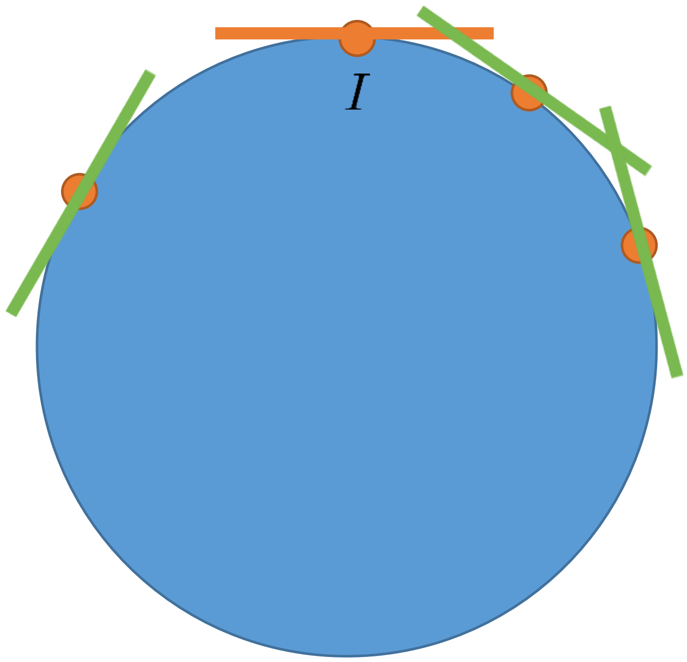
The Lie Algebra is a **vector space**, along with a binary operation called the **Lie Bracket**

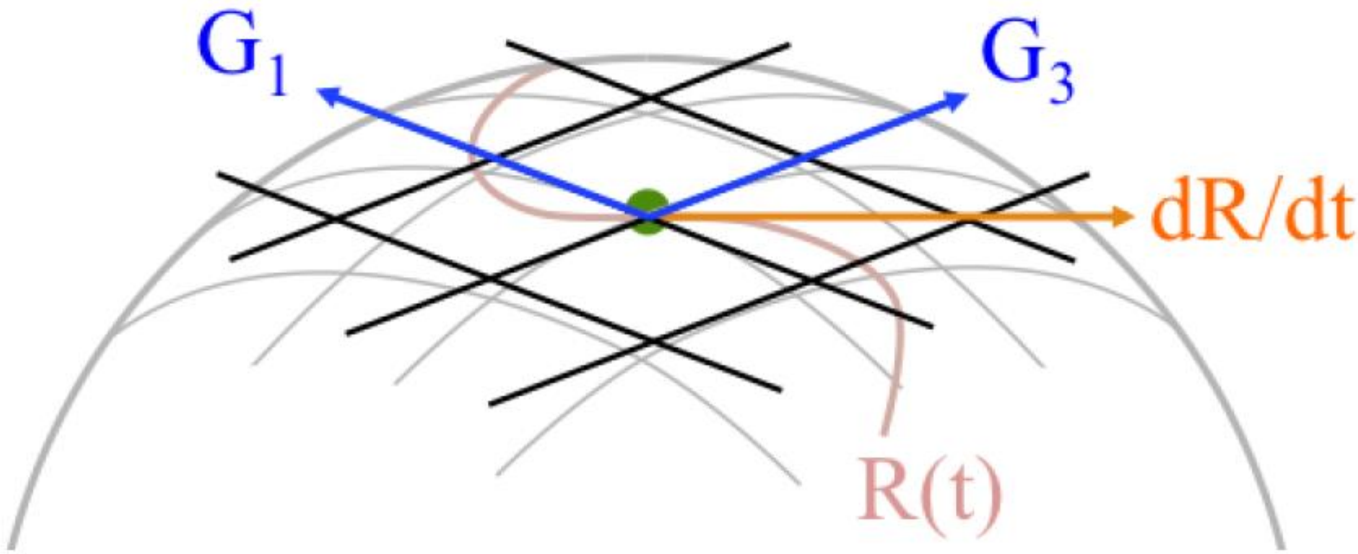
$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$$

$$[a, b] : ab - ba$$

Must satisfy full list of axioms associated with Lie Bracket (https://en.wikipedia.org/wiki/Lie_algebra)

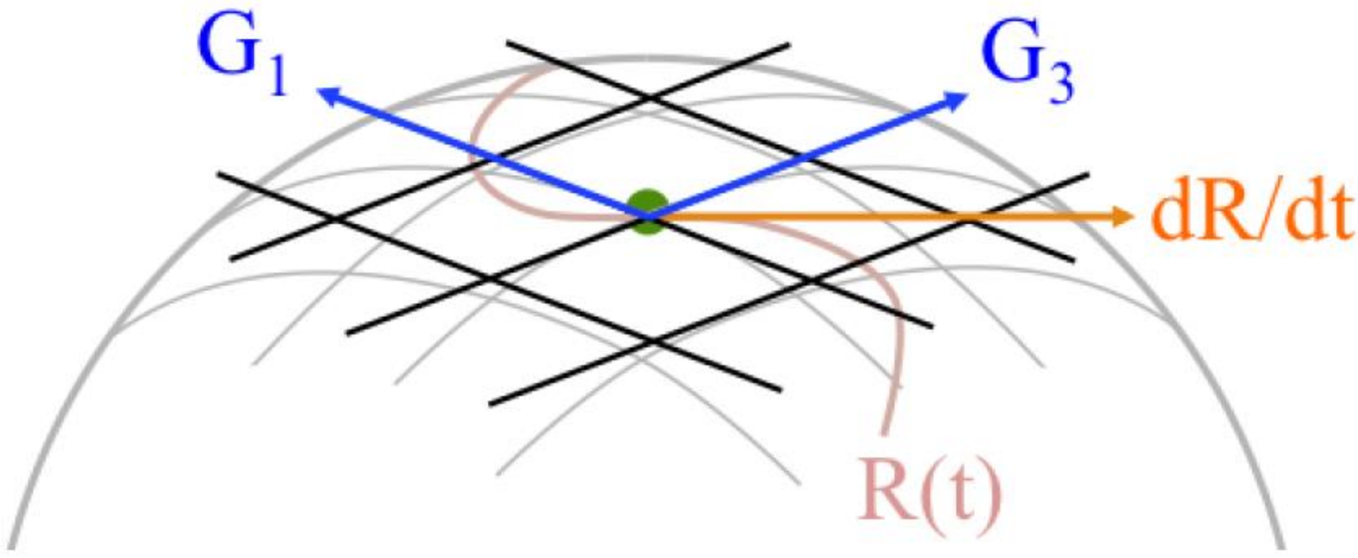
The tangent space at the identity element (Lie Algebra) is **isomorphic** to the tangent space of the other group elements.





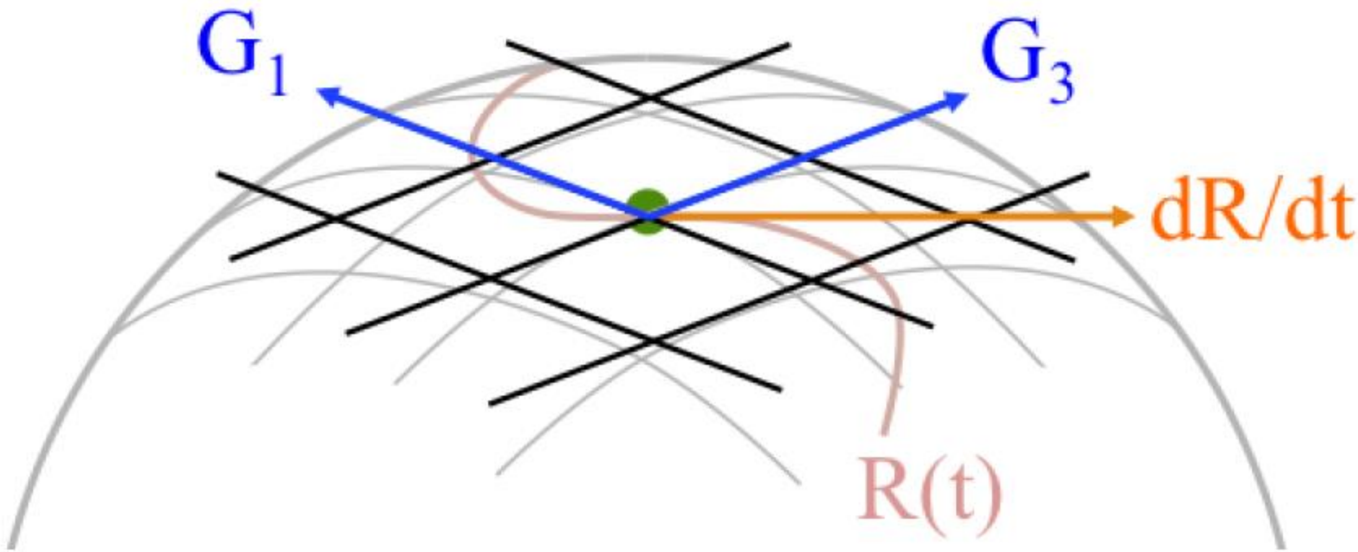
Define a tangent space element at the identity:

$$\omega \in \mathfrak{g}$$



The differential equation that relates the tangent spaces:

$$\frac{dR}{dt} = R\omega$$



Solve the differential eqn with initial condition $R(0) = I$

$$R(t) = e^{t\omega}$$

We call this relationship between the Lie Algebra and Lie group the **exponential map.** For SO(3), when t=1

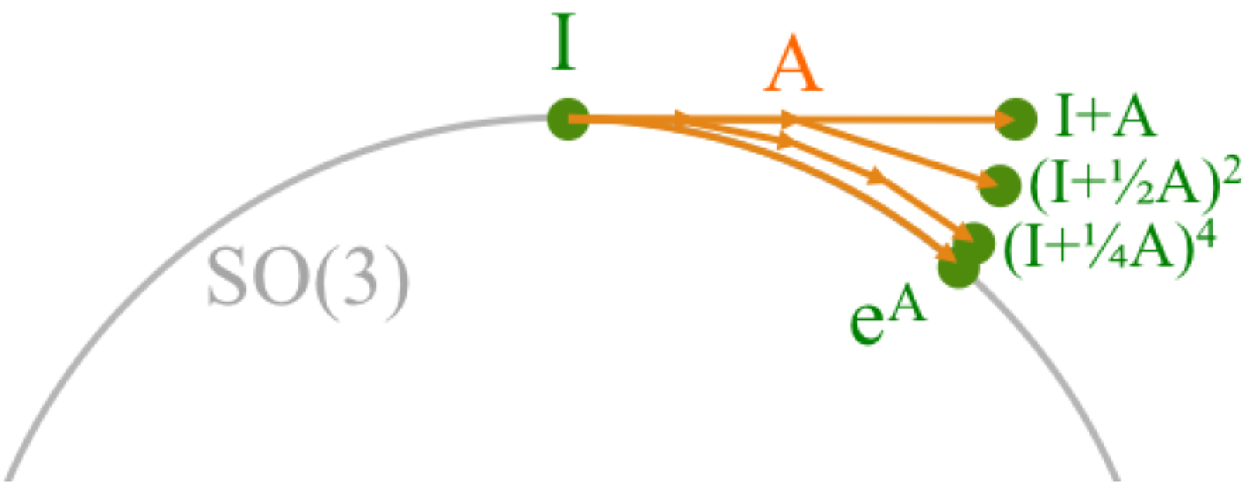
$$R(t) = e^{t\omega}$$

$$\mathbf{so}(3) \mapsto \mathbb{SO}(3)$$
$$\omega \mapsto e^{\omega}$$

Visual example of exponential map

$$e^A = \lim_{n \to \infty} \left(I + \frac{1}{n} A \right)^n$$

As n grows, quantity inside brackets becomes member of SO(3). Multiplication n times also gives member of SO(3). Successive approximations:



For SO(3), the Lie Algebra is the set of all skew-symmetric matrices

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge\right] = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} = aG_1 + bG_2 + cG_3$$

This gives us a very nice parameterization for SO(3), using an element in $\mathbb{R}^3$ to represent an element in SO(3).

$$R = e^{[\mathbf{v}\wedge]}$$

Exponential Map for SO(3) has a closed form solution, called the Rodrigues formula.

$$\omega : \text{axis of rotation}$$
$$\theta = ||\omega|| : \text{magnitude of rotation}$$

$$\exp(\boldsymbol{\omega}_\times) = \mathbf{I} + \left(\frac{\sin\theta}{\theta}\right)\boldsymbol{\omega}_\times + \left(\frac{1-\cos\theta}{\theta^2}\right)\boldsymbol{\omega}_\times^2$$

We can also invert the exponential map using the **logarithmic map**.

$$\ln(\mathbf{R}) : \mathbb{SO}(3) \mapsto \mathbf{so}(3)$$

$$\theta = \arccos\left(\frac{\mathrm{tr}(\mathbf{R})-1}{2}\right)$$
$$\ln(\mathbf{R}) = \frac{\theta}{2\sin\theta}\cdot(\mathbf{R}-\mathbf{R}^T)$$

We can use a similar derivation to get the generators of SE(3)

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G_6 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \begin{pmatrix} \mathbf{u} & \boldsymbol{\omega} \end{pmatrix}^T &\in \mathbb{R}^6 \\ \mathbf{u}_1 G_1 + \mathbf{u}_2 G_2 + \mathbf{u}_3 G_3 + \boldsymbol{\omega}_1 G_4 + \boldsymbol{\omega}_2 G_5 + \boldsymbol{\omega}_3 G_6 &\in \mathrm{se}(3) \end{aligned}$$

G1,G2,G3 are generators for the translations, G4,G5,G6 are the generators for the rotation.

The exponential map for SE(3) also has a closed form:

$$
\begin{aligned}
\mathbf{u}, \boldsymbol{\omega} &\in \mathbb{R}^3 \\
\theta &= \sqrt{\boldsymbol{\omega}^T \boldsymbol{\omega}} \\
A &= \frac{\sin\theta}{\theta} \\
B &= \frac{1-\cos\theta}{\theta^2} \\
C &= \frac{1-A}{\theta^2} \\
\mathbf{R} &= \mathbf{I} + A\boldsymbol{\omega}_\times + B\boldsymbol{\omega}_\times^2 \\
\mathbf{V} &= \mathbf{I} + B\boldsymbol{\omega}_\times + C\boldsymbol{\omega}_\times^2 \\
\exp\begin{pmatrix} \mathbf{u} \\ \boldsymbol{\omega} \end{pmatrix} &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{V}\mathbf{u} \\ \hline \mathbf{0} & 1 \end{array}\right)
\end{aligned}
$$

The logarithmic map for SE(3) also has a closed form:

$$\mathbf{T} = \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right)$$

$$\theta = \arccos\left(\frac{\mathrm{tr}(\mathbf{R}) - 1}{2}\right)$$

$$\omega_\times = \ln(\mathbf{R}) = \frac{\theta}{2\sin\theta} \cdot (\mathbf{R} - \mathbf{R}^T)$$

$$\ln(\mathbf{T}) : \mathbb{SE}(3) \mapsto \mathbf{se}(3)$$

$$\mathbf{T} \mapsto [\mathbf{u}, \omega]$$

$$\mathbf{V}^{-1} = \mathbf{I} - \frac{1}{2}\omega_\times + \frac{1}{\theta^2}\left(1 - \frac{A}{2B}\right)\omega_\times^2$$

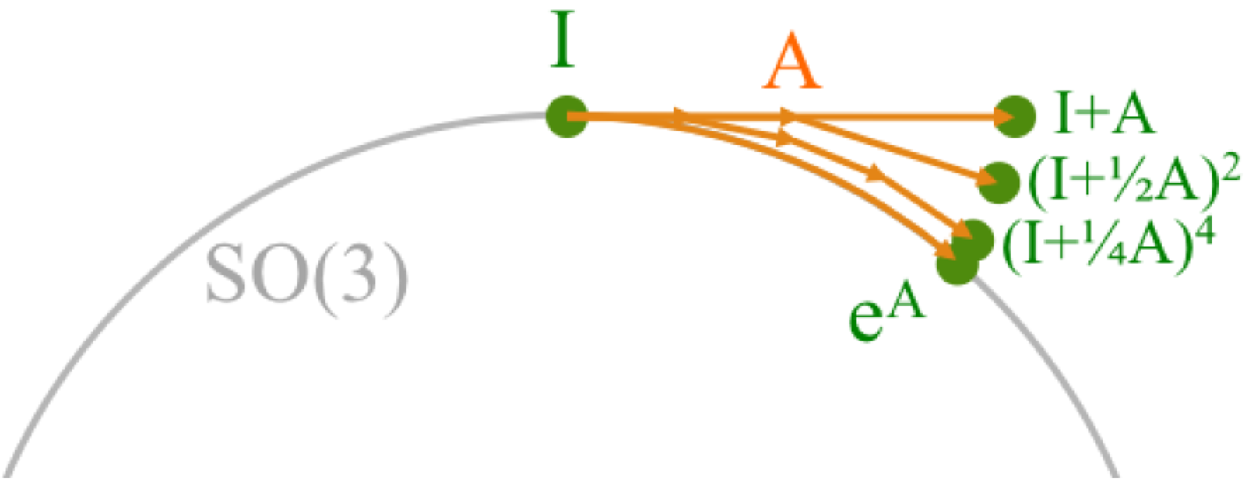
$$\mathbf{u} = \mathbf{V}^{-1} \cdot \mathbf{t}$$

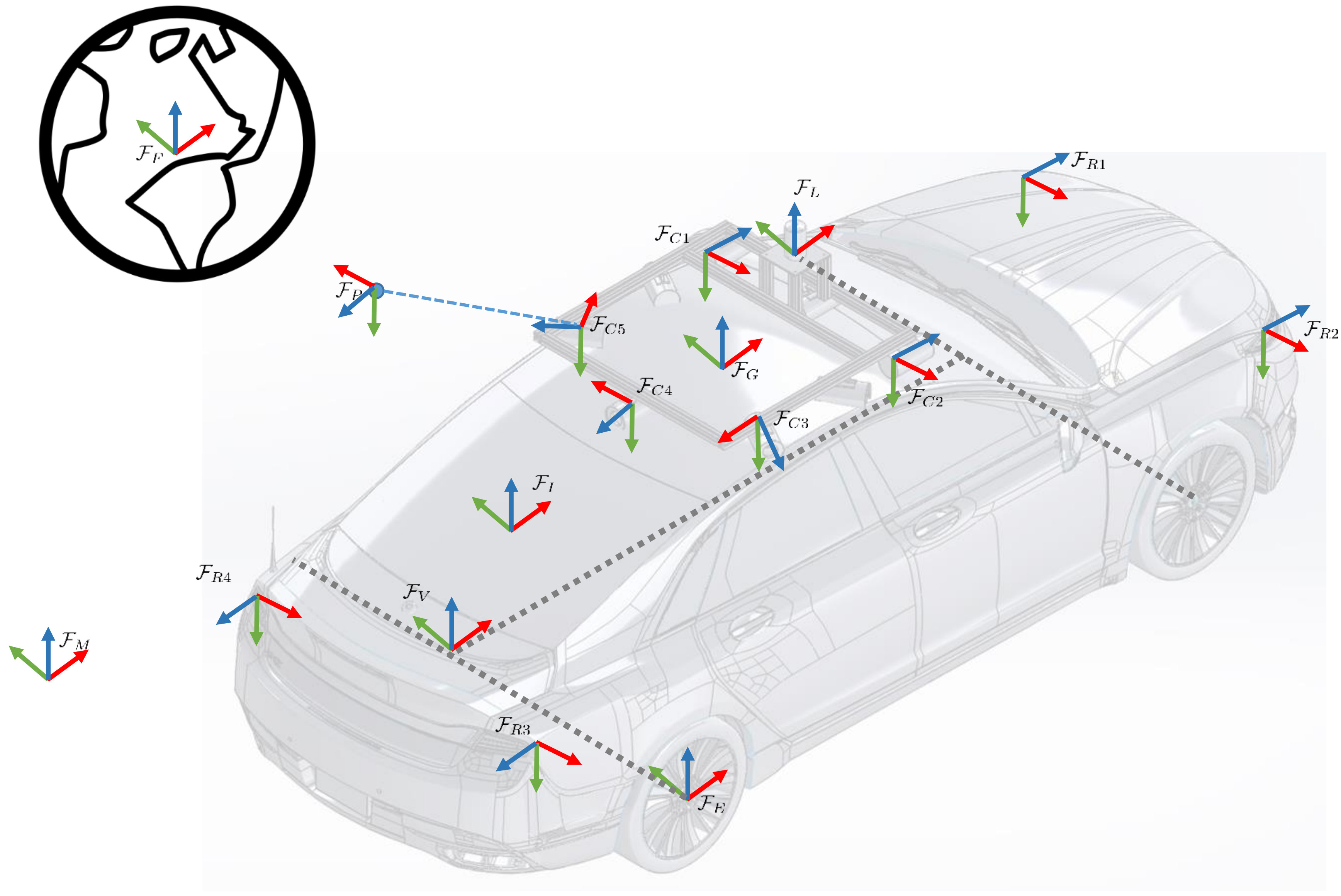
	Rotation	Transformation
Matrix	3x3 matrix R	4x4 matrix $\mathbf{T} = \left(\begin{array}{c\|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array} \right)$
Lie Group	SO(3)	SE(3)
Lie Algebra	so(3)	se(3)
Tangent vectors	"angular velocities" $\omega_\times$	"twist" $\mathbf{A}(\mathbf{v}) = \begin{pmatrix} [\omega]_\times & \mathbf{t} \\ 0 & 0 \end{pmatrix}$

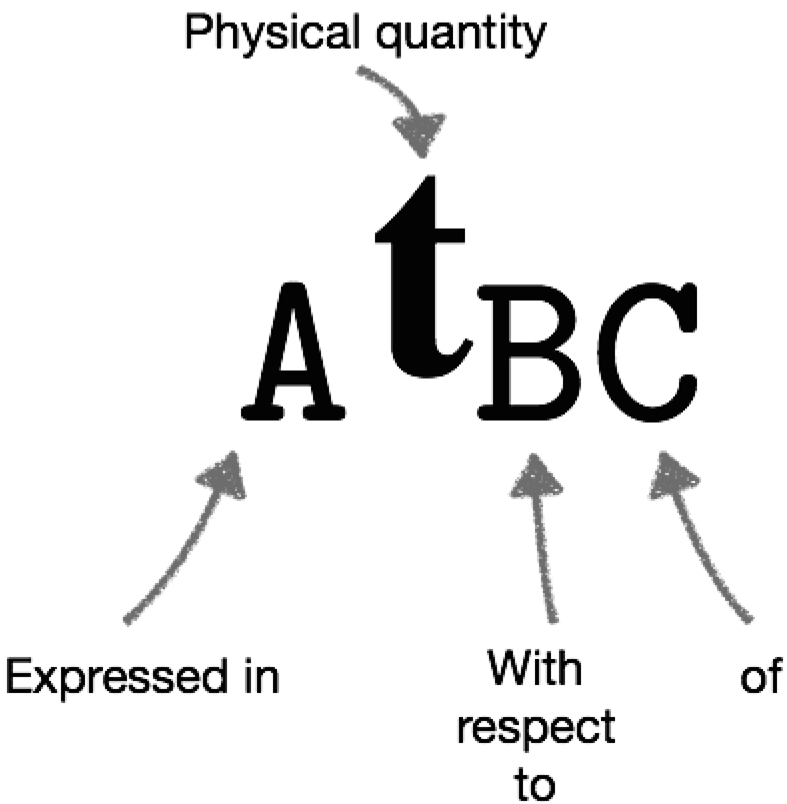
- DONE!

- Still investigating in more detail
- exp map is **not** one-to-one, will map rotations in multiples of 2π to same group element
  - Also need to be careful when rotation magnitude is close to zero

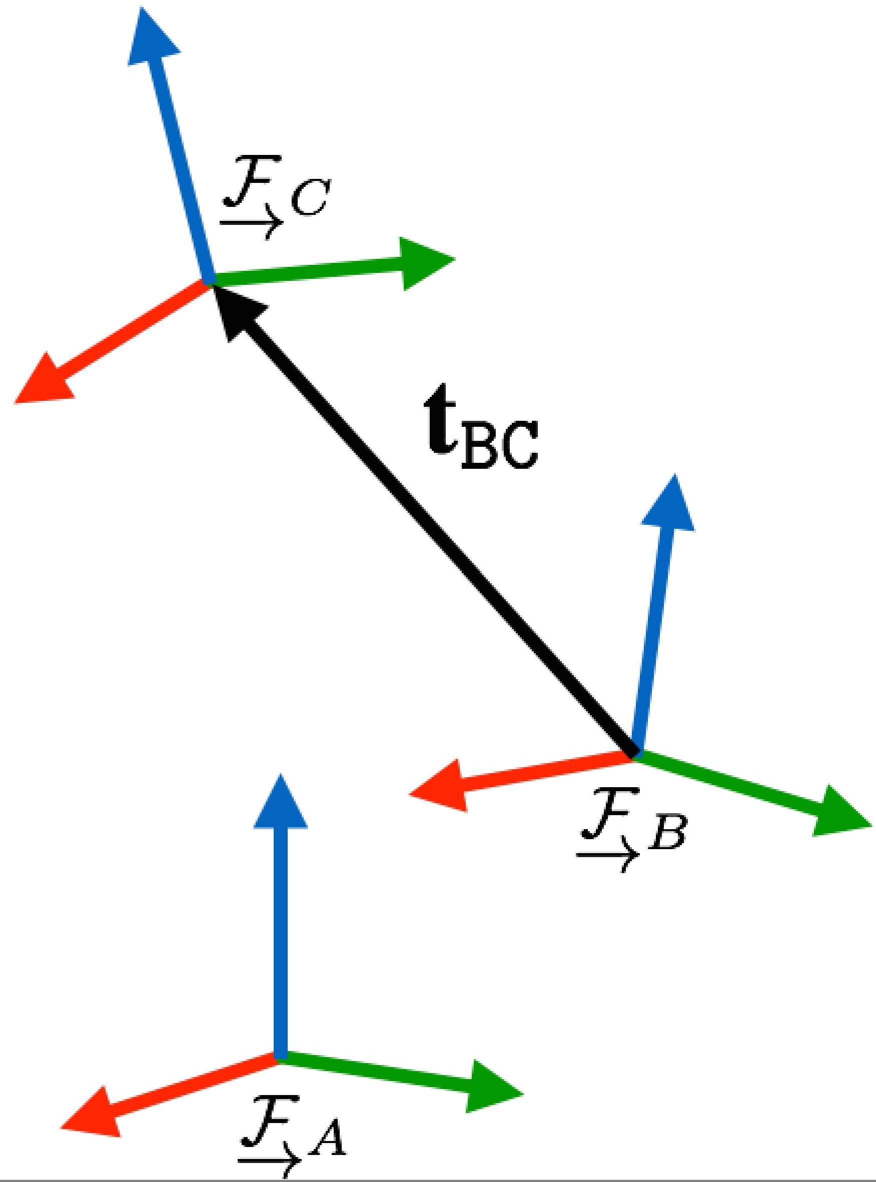
$$\exp\left(\boldsymbol{\omega}_\times\right) = \mathbf{I} + \left(\frac{\sin\theta}{\theta}\right)\boldsymbol{\omega}_\times + \left(\frac{1-\cos\theta}{\theta^2}\right)\boldsymbol{\omega}_\times^2$$

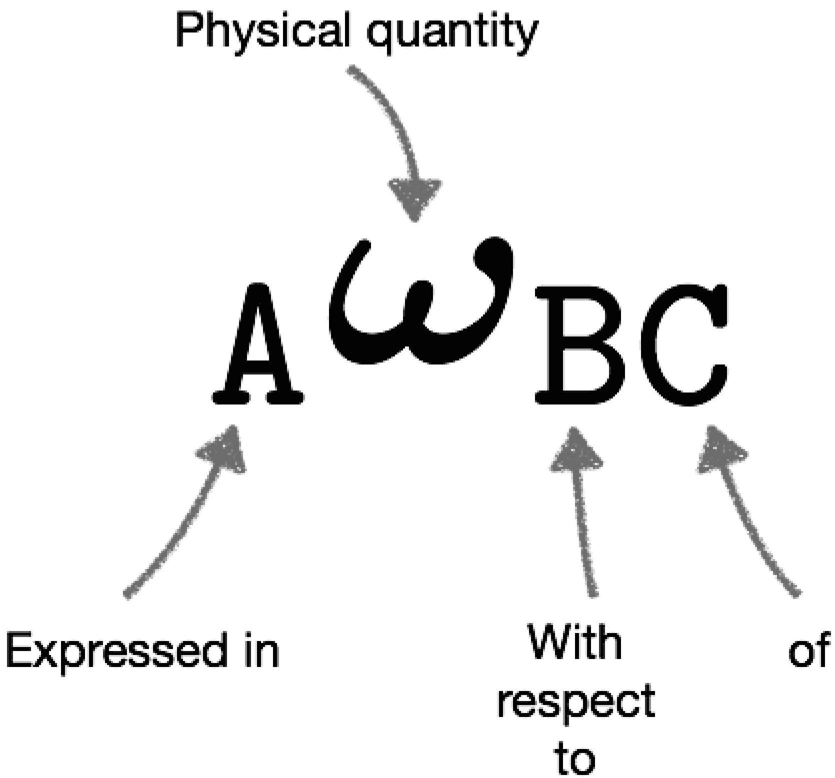



$\mathcal{F}_F$
$\mathcal{F}_{R1}$
$\mathcal{F}_L$
$\mathcal{F}_{C1}$
$\mathcal{F}_P$
$\mathcal{F}_{C5}$
$\mathcal{F}_G$
$\mathcal{F}_{R2}$
$\mathcal{F}_{C4}$
$\mathcal{F}_{C2}$
$\mathcal{F}_{C3}$
$\mathcal{F}_I$
$\mathcal{F}_{R4}$
$\mathcal{F}_V$
$\mathcal{F}_M$
$\mathcal{F}_{R3}$
$\mathcal{F}_E$

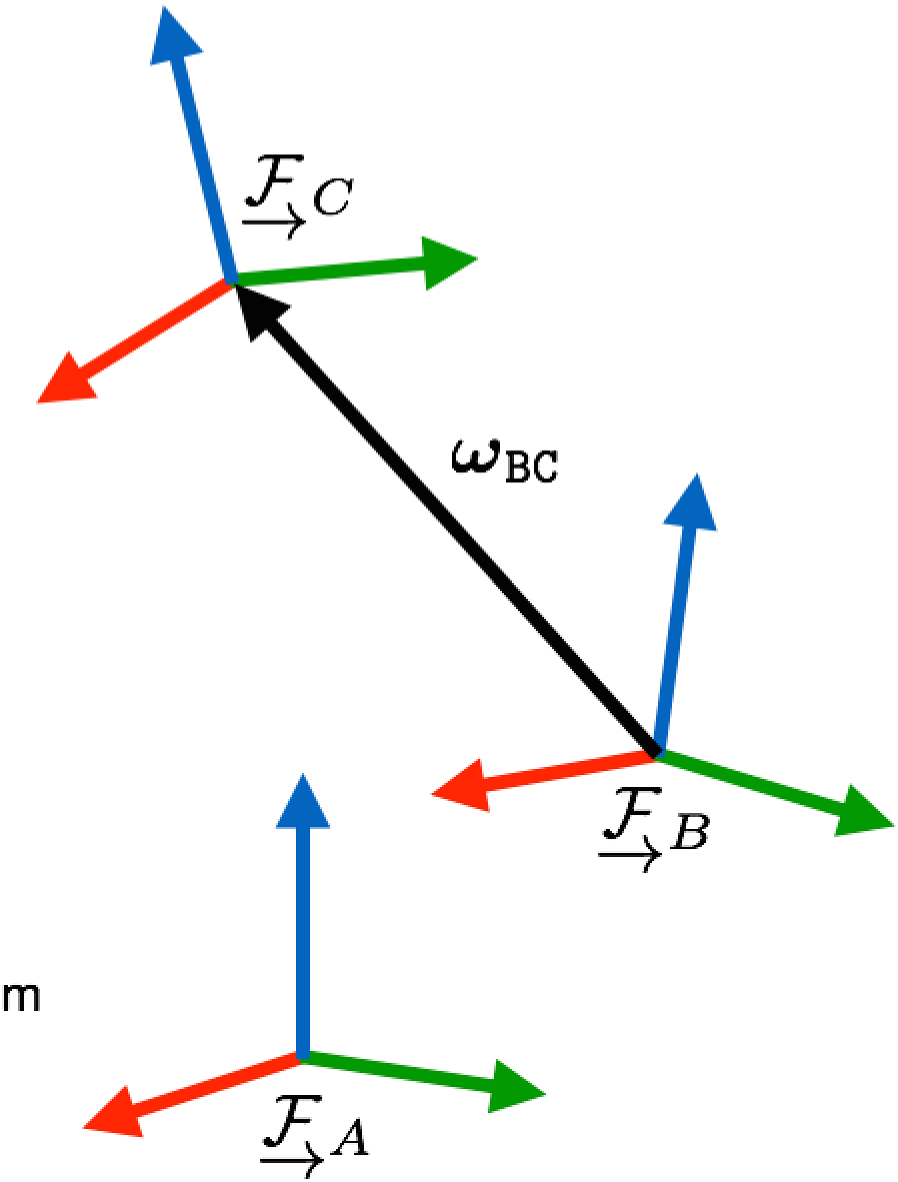


"The vector from B to C expressed in A"





"The angular velocity of frame C as seen from frame B, expressed in frame A"



- Be very clear in terms of rotation direction

The resulting rotation matrix, $\mathbf{C}_{WB}$, represents the orientation of the robot body frame, $\underrightarrow{\mathcal{F}}_{B}$, with respect to the world frame, $\underrightarrow{\mathcal{F}}_{W}$, such that a vector expressed in the body frame, ${}_{B}\mathbf{v}$, can be rotated into the world frame by

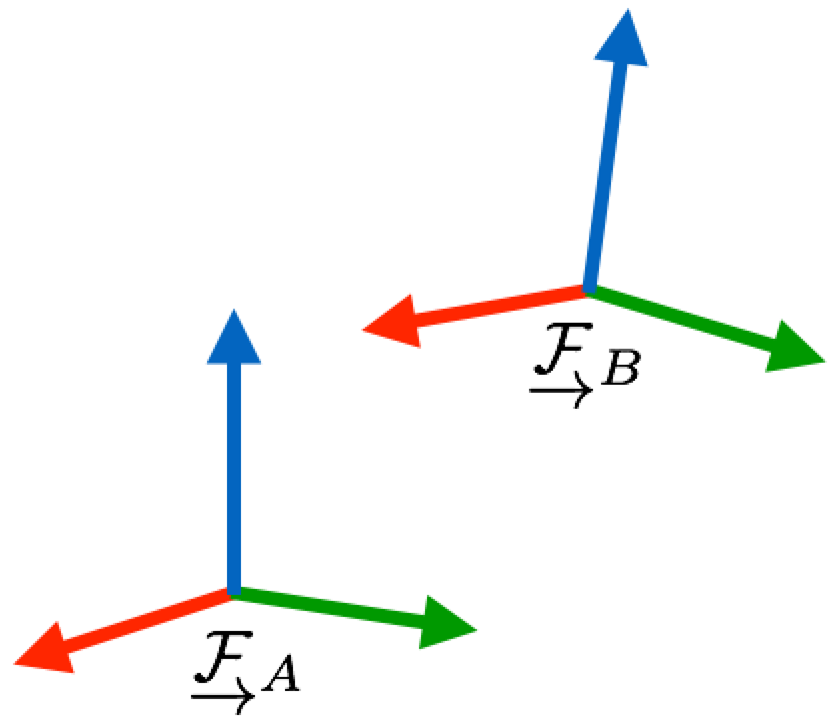
$$ {}_{W}\mathbf{v} = \mathbf{C}_{WB}\,{}_{B}\mathbf{v}. \tag{1} $$

"The orientation/attitude of B with respect to A"

"Rotates vectors from frame B into frame A"

$$\mathbf{C}_{AB}$$

With respect to (A) — of (B)



$$_{A}\mathbf{v}_{BC} = \mathbf{C}_{AB}\,{}_{B}\mathbf{v}_{BC}$$

- Be very clear about the pose transformation direction

The resulting transformation matrix, $\boldsymbol{T}_{\mathtt{WB}}$, represents the pose of the robot body frame, $\underrightarrow{\mathcal{F}}_{\mathtt{B}}$, with respect to the world frame, $\underrightarrow{\mathcal{F}}_{\mathtt{W}}$, such that a point expressed in the body frame, ${}_{\mathtt{B}}\mathbf{p}$, can be transformed into the world frame by

$$ {}_{\mathtt{W}}\boldsymbol{p} = \boldsymbol{T}_{\mathtt{WB}}\,{}_{\mathtt{B}}\boldsymbol{p}. \tag{1} $$

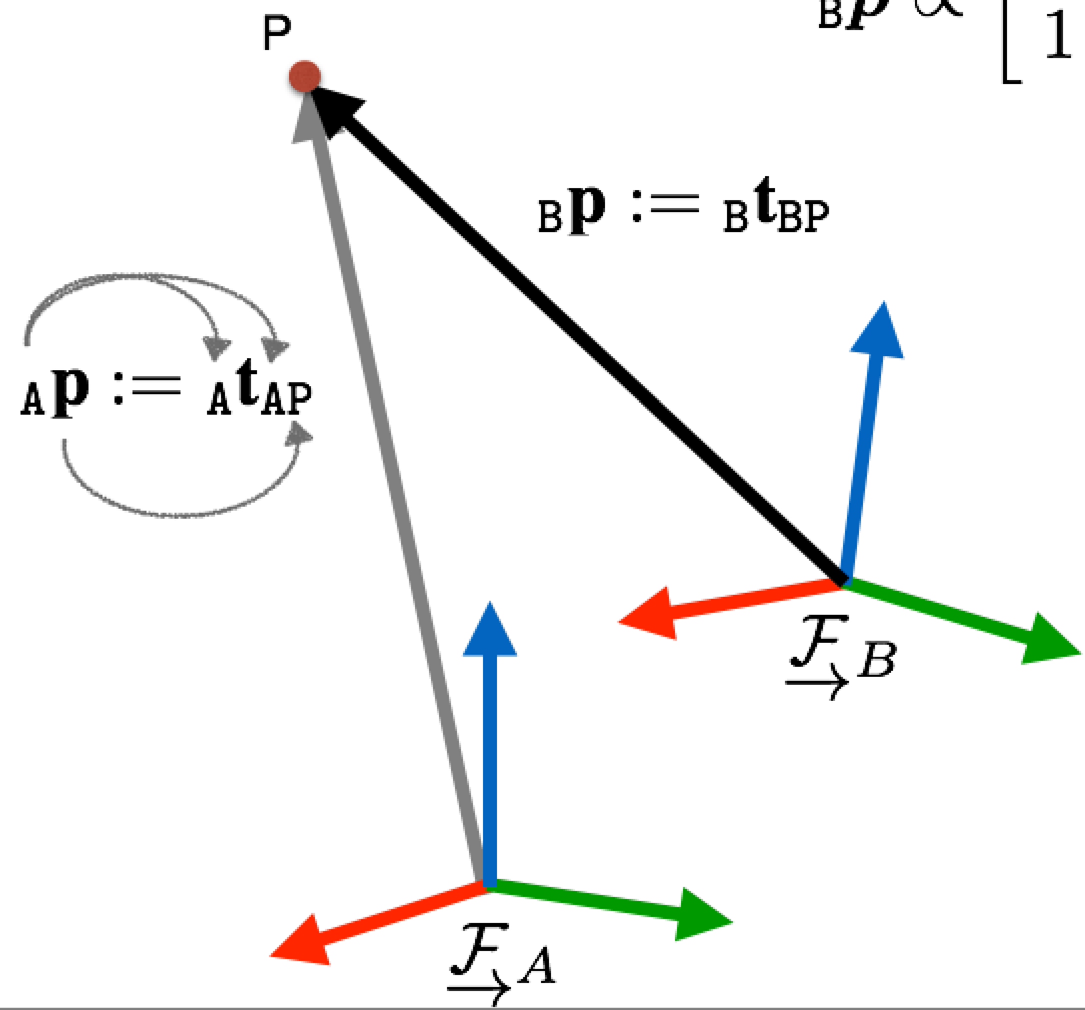
"The pose of B with respect to A"

"Transforms point from frame B into frame A"

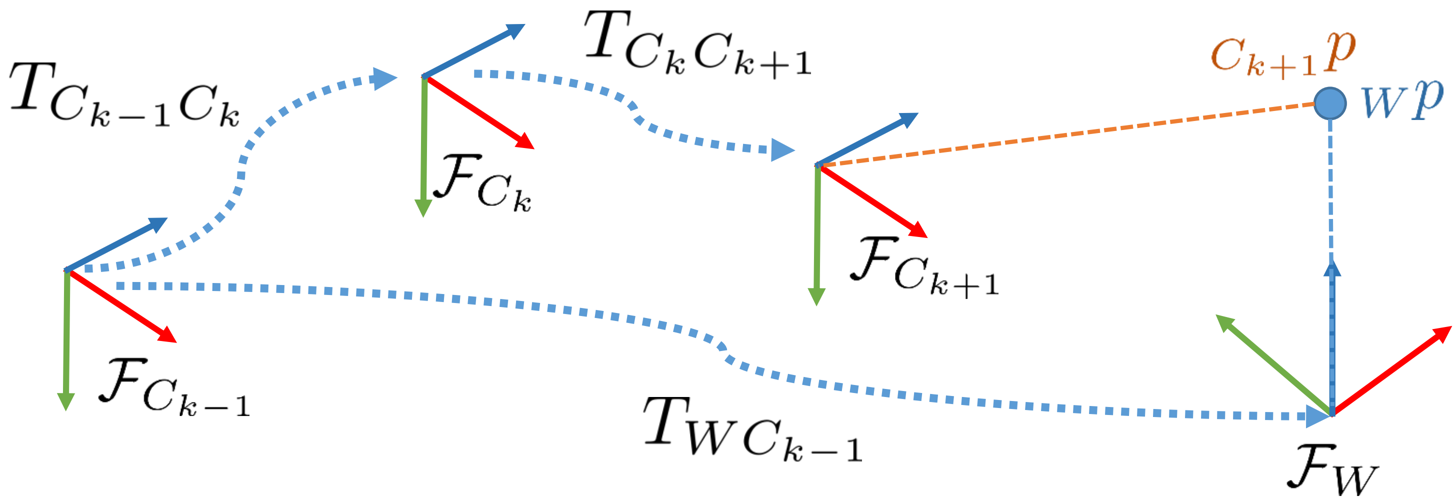
$${}_{\mathrm{B}}\boldsymbol{p} \propto \begin{bmatrix} {}_{\mathrm{B}}\mathbf{p} \\ 1 \end{bmatrix}$$

$$\boldsymbol{T}_{\mathrm{AB}}$$

into (A) — transforms points from (B)



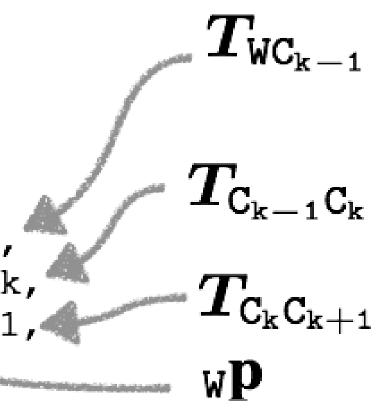
$${}_{\mathrm{A}}\boldsymbol{p} = \boldsymbol{T}_{\mathrm{AB}\,\mathrm{B}}\boldsymbol{p}$$




```
/// Coordinate frames in this function:
///    - C : The camera frame, indexed by time, k.
///    - W : The world frame.
Point pointToCamera( const Transformation& T_W_Ckm1,
                     const Transformation& T_Ckm1_Ck,
                     const Transformation& T_Ck_Ckp1,
                     const Point& W_p ) {

  Transformation T_Ckp1_W = (T_W_Ckm1 * T_Ckm1_Ck * T_Ck_Ckp1).inverse();
  return T_Ckp1_W * W_p

}
```



Frame	Description	Kinematics Tag
Camera	Camera Lens Optical Center	C
Point	Landmark (Point) Frame	P
Vehicle	Centre of Back Axle	V
IMU	IMU Origin	I
GPS	Antenna Phase Center	G
Lidar	Local Lidar Point Frame	L
Encoder	About axis of rotation	E
Radar	Radar Center	R
ECEF	Earth Centered Earth Fixed	F
Map	Localization Map Frame	M

- In depth derivations of the closed forms for exp and log maps on SO(3) and SE(3)
- Adjoint mapping and representations
- Defining “distance” between group elements
- Differential Calculus on SO(3) and SE(3)
- Implementation using wave::kinematics
- Building residual terms using wave::kinematics