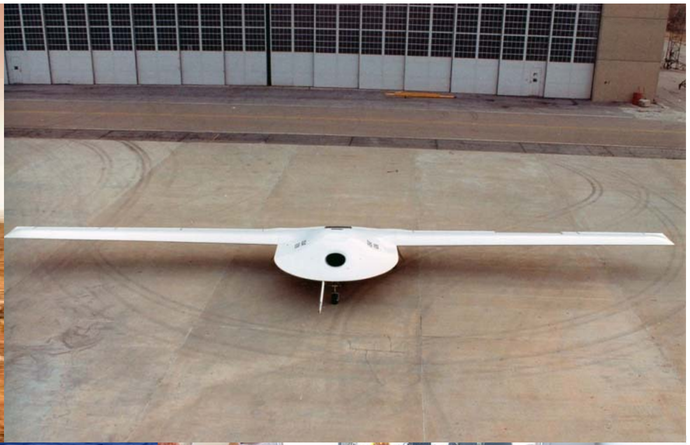
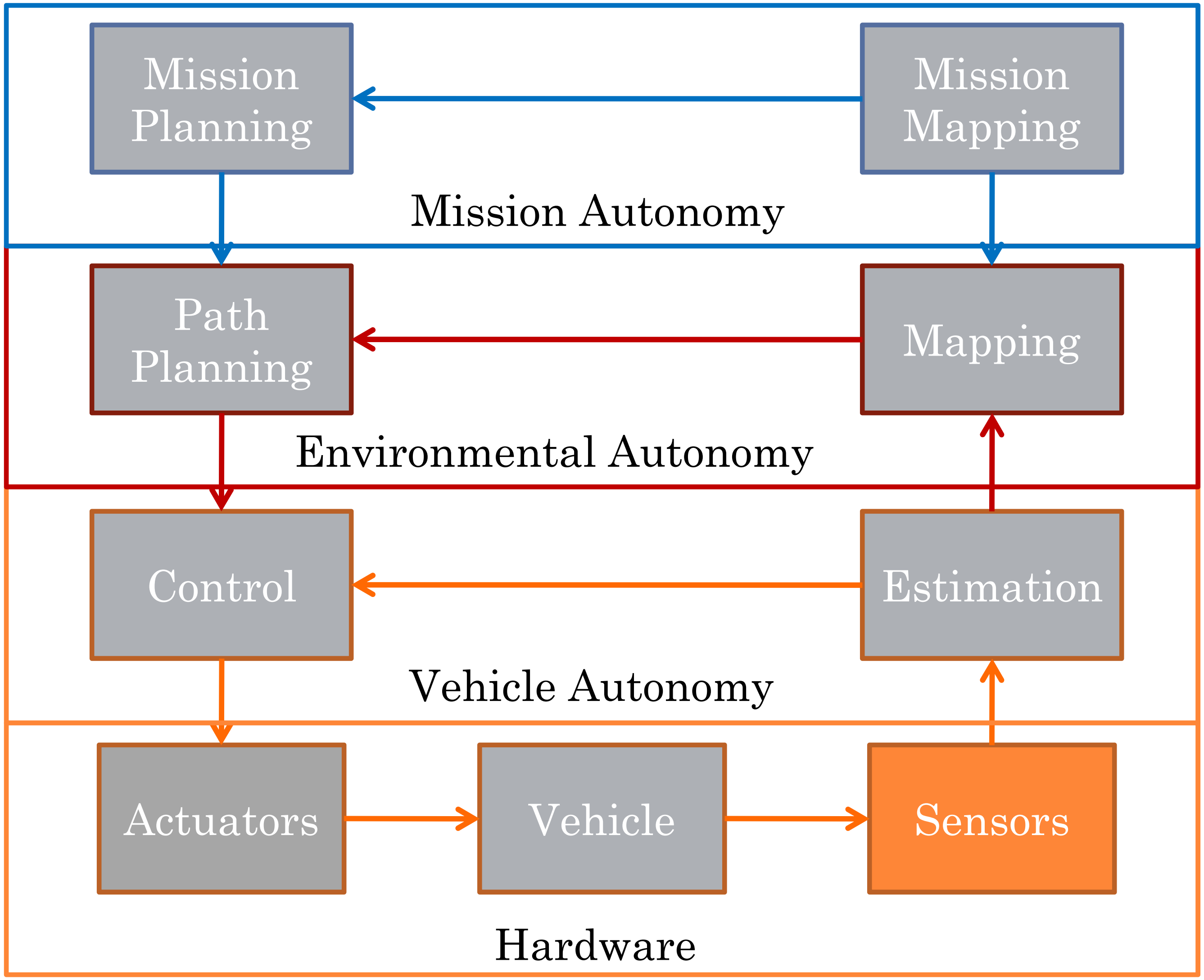





ME 597: AUTONOMOUS MOBILE ROBOTICS
SECTION 4 – MEASUREMENT

Prof. Steven Waslander

COMPONENTS



Mission Planning
Mission Mapping
Mission Autonomy
Path Planning
Mapping
Environmental Autonomy
Control
Estimation
Vehicle Autonomy
Actuators
Vehicle
Sensors
Hardware

Outline

- Measurement Models
- Sensor Characteristics
- Sensors
  - Contact Sensors
  - Inertial Sensors
  - Range Sensors
  - Position Sensors
  - Vision

Measurements

Definition: A ***measurement*** provides information about the state of the system.

- Sensors are those components of the robot that provide measurements

- Measurements can be referred to as an output
  - The term output is usually associated with elements of the state that are to be controlled

- The second half of this section is devoted to sensor examples and their characteristics

MEASUREMENT MODEL

- The discrete time measurement model is defined as

$$y_t = g(x_t, \delta_t)$$

  - x is the state vector at time t
  - $g(x_t)$ is the measurement model
  - δ_t is the noise on the measurements

- Combined with the motion model, the full system is

$$x_t = f(x_{t-1}, u_t, \varepsilon_t)$$

$$y_t = g(x_t, \delta_t)$$

Measurement Model

- Linear Time-Invariant (LTI) State Space Measurement Model

$$y_t = Cx_t + Du_t + \delta_t, \qquad \delta_t \sim N(0, Q_t)$$

- Input is explicitly included in measurement model in standard derivation (rarely used!)

- (A,B,C,D) defines a complete LTI state space system

$$x_t = Ax_{t-1} + Bu_t + \varepsilon_t$$
$$y_t = Cx_t + Du_t + \delta_t$$

Measurement Model

- For estimation algorithms, we'll use probabilistic definition

$$p(y_t \mid x_t)$$

  - Probability of a measurement y occurring at time t, given the state at time t is x.
  - Assume no explicit dependence of measurements on current input
    - Complete state makes this a reasonable assumption
    - Not assumed in standard LTI state space systems
  - Analogous definition to the motion model

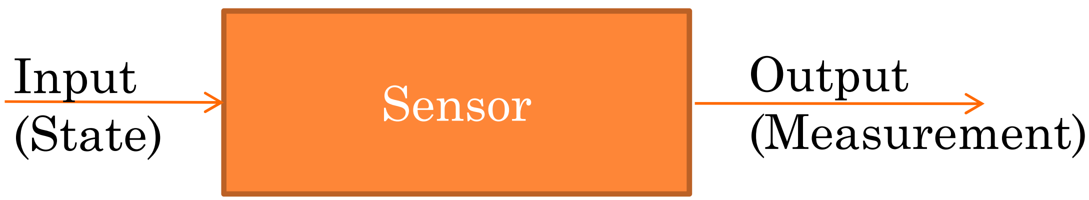
$$p(x_t \mid x_{t-1}, u_t)$$

Outline

- Measurement Models
- Sensor Characteristics
- Sensors
  - Contact Sensors
  - Inertial Sensors
  - Range Sensors
  - Position Sensors
  - Vision

Sensor Characteristics

- Sensor as a system



- Passive/Active
  - Passive sensors measure energy coming to the sensor from the environment (eg. vision)
  - Active sensors emit energy and measure the reaction of the environment (eg. SODAR sound pulse)

Sensor Characteristics

- Range
  - Lower and upper limits of sensor inputs
    - Set of all possible states that can be measured

- Dynamic Range
  - Defined by spread between lowest and highest inputs, often recorded in decibels (dB)

$$DR = 10\log\left(\frac{x_{\max}}{x_{\min}}\right)$$

  - Important to understand what happens when a sensor receives an input that is out of range
    - e.g. image pixel, LIDAR

- Full Scale
  - Lower and upper limits of sensor output values
    - Difference between min and max output

SENSOR CHARACTERISTICS

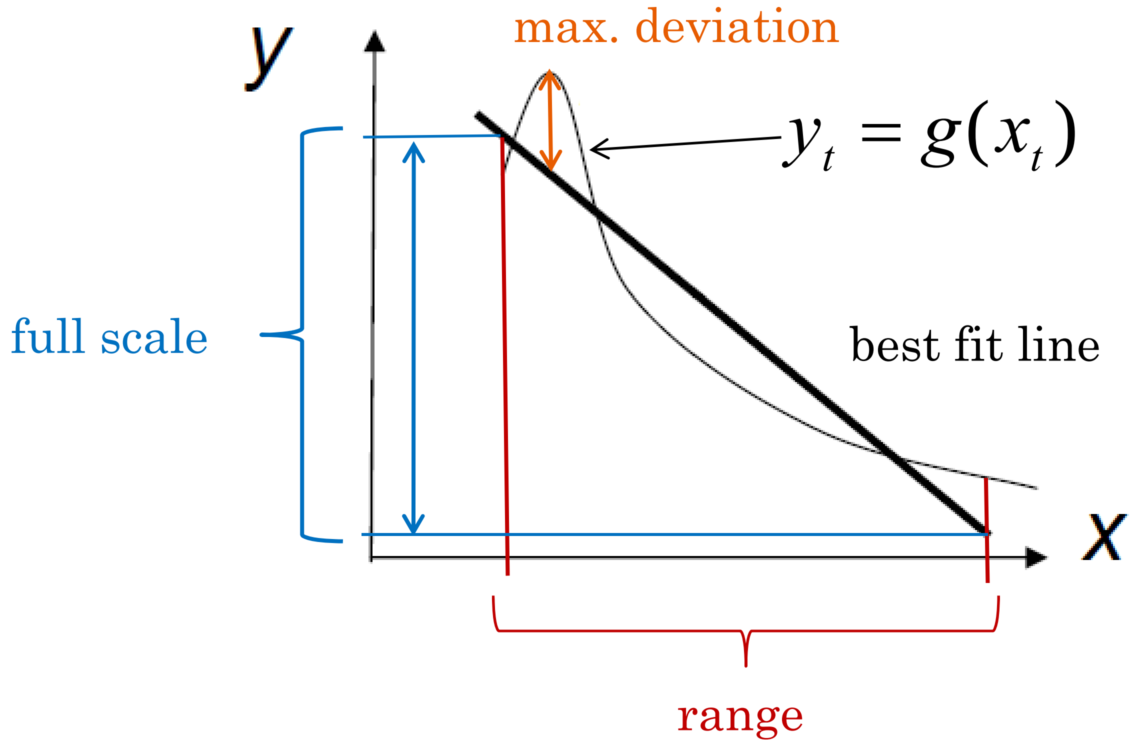
- Linearity
  - A linear relationship between input and output is desirable for sensors
  - Linearity defines how linear this relationship is
  - Often quoted as 5% non-linearity
    - Percentage of maximum deviation from best fit line over full scale of sensor outputs

$$\%NL = 100\% \left(\frac{\text{max deviation}}{\text{full scale}} \right)$$

  - Can often be calibrated for if measurement signal is available digitally

Sensor Characteristics

- Linearity Example
  - Consider an IR sensor ranging an obstacle



  - Linear is not a great approximation in this case

Sensor Characteristics

- Bandwidth/Frequency
  - Ability to track continuous sinusoidal input (bandwidth, continuous measurements)
  - The rate of sensor measurement updates (frequency, discrete measurements)
  - Higher frequencies are needed for inner loop control (fast response), lower frequencies for mapping, planning

- Resolution
  - Minimum separation between two measurement values
  - For analog sensors, usually A/D resolution

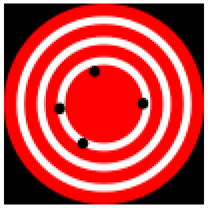
SENSOR CHARACTERISTICS

- Accuracy
  - The difference between the sensor's measured value (m) and the true value (v) (e.g. 95% accurate)

$$a = 1 - \frac{|m - v|}{v}$$

- Precision
  - The reproducibility of results

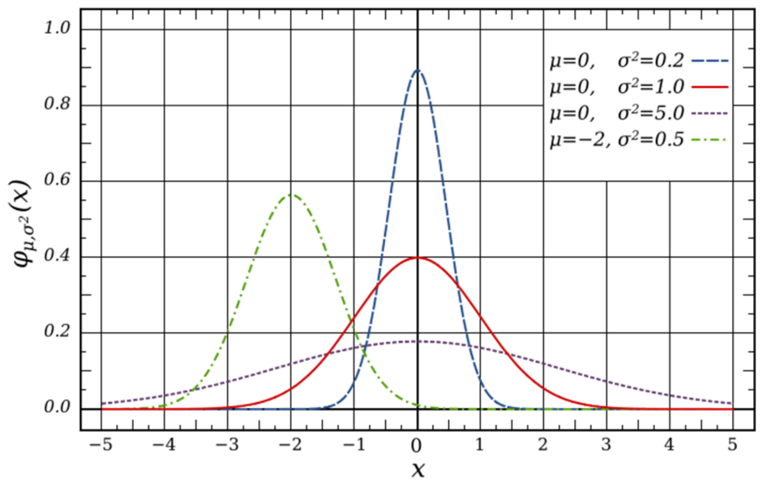
$$\text{precision} = \frac{\text{output range}}{\text{standard deviation}}$$



Accuracy vs Precision
High or Low?

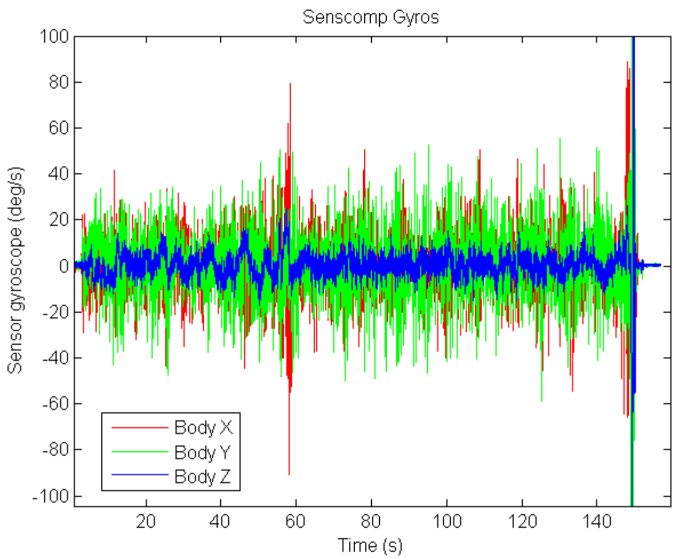
SENSOR CHARACTERISTICS

- Systematic Error
  - Deterministic
  - Caused by factors that can be modeled (e.g. camera distortions, gyroscope biases)
- Random Error
  - Non-deterministic
  - Can be modeled as random variables with known probability distributions, often Gaussian



Sensor Noise Model Identification

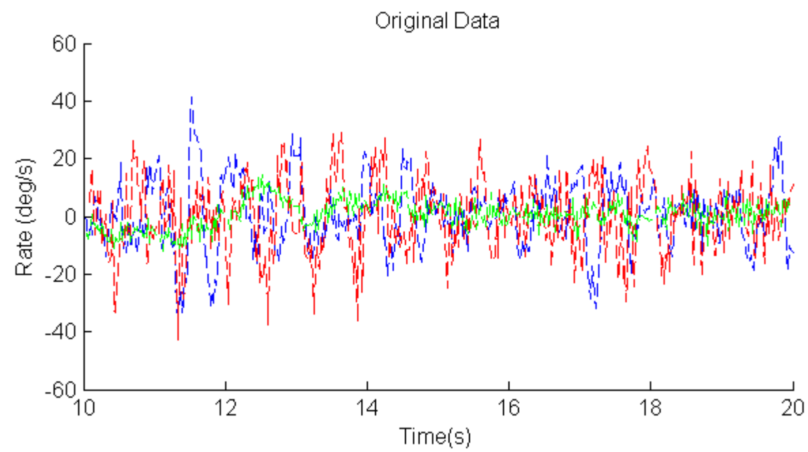
- Measurement Model Identification
  - Matching a Normal distribution to gyro data
    - Assume measurement independence
    - Assume no underlying motion
      - Hover flight, not really valid
    - Attempt to remove motion
      - e.g. Moving average filter
    - Generate normally distributed samples and compare

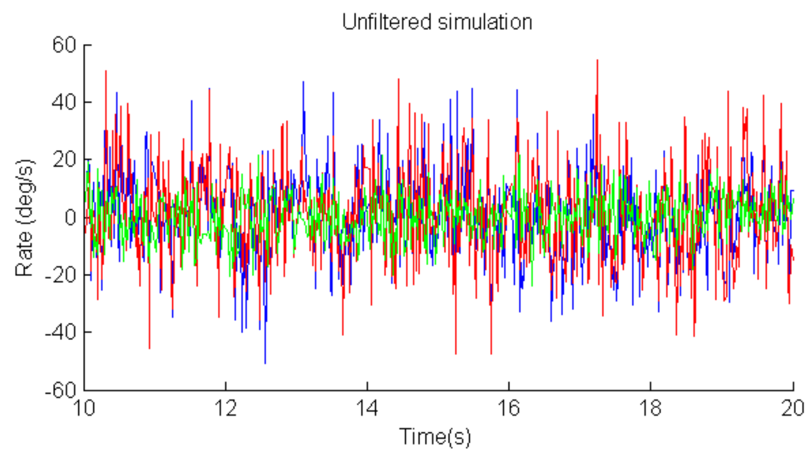


Sensor Noise Model Identification

- Example
  - Gyro results

$$\mathrm{cov} = \begin{bmatrix} 224 & 33 & -7 \\ 33 & 289 & -9 \\ -7 & -9 & 66 \end{bmatrix}$$





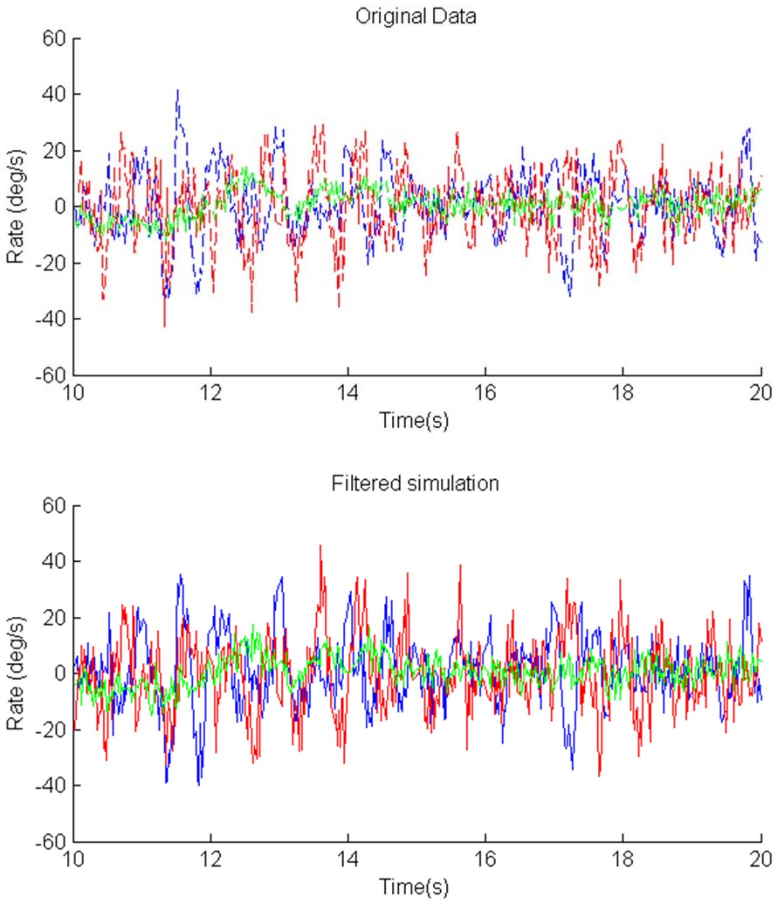
- Measurement variance is too large
- Includes vehicle motion (unknown)

Sensor Noise Model Identification

- Example
  - Gyro results
    - Filter motion
    - 3-step moving average
    - Simulate noise with

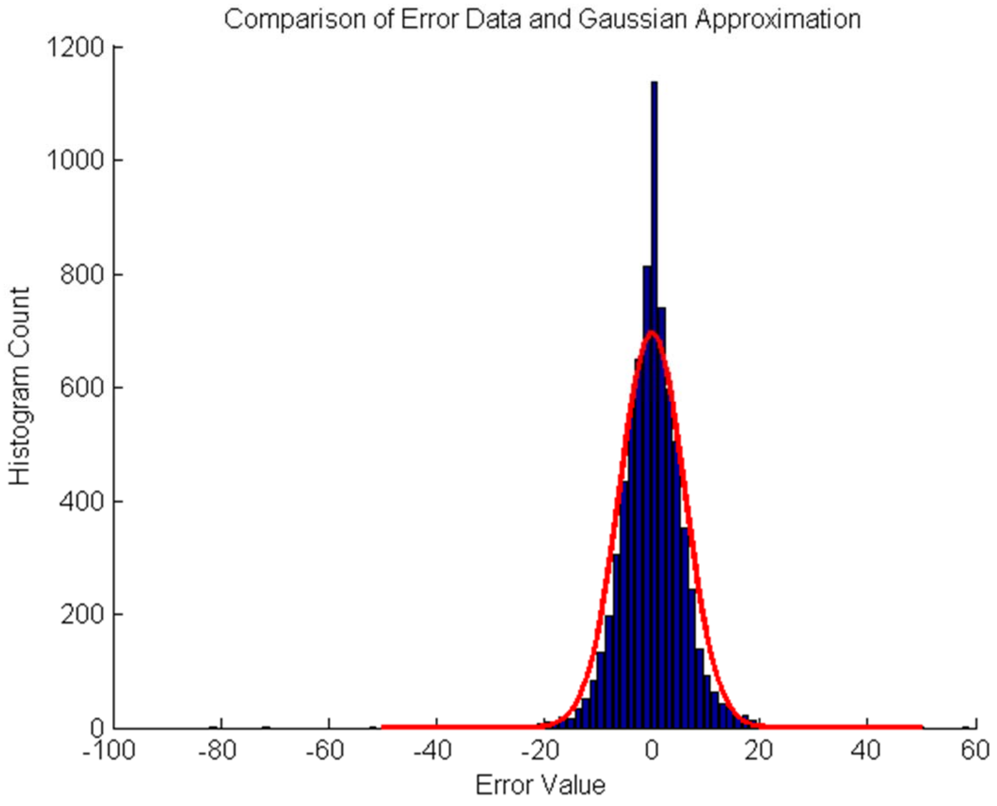
$$\mathrm{cov} = \begin{bmatrix} 35 & -0.5 & -1.5 \\ -0.5 & 71 & -1.5 \\ -1.5 & -1.5 & 10 \end{bmatrix}$$

    - Add noise to filtered motion



Sensor Noise Model Identification

- Comparison of the x-axis gyro errors and resulting Gaussian
  - Errors are after removal of filtered rate

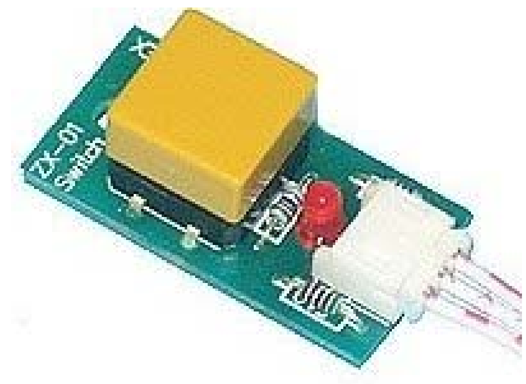


Outline

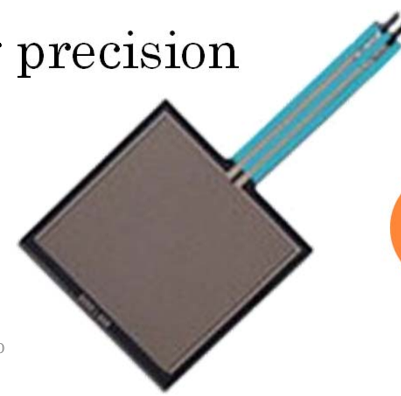
- Measurement Models
- Sensor Characteristics
- Sensors
  - Contact Sensors
  - Inertial Sensors
  - Range Sensors
  - Position Sensors
  - Vision

CONTACT SENSING

- **Push button**
  - Simple. Digital output: 0 or 1
  - Can be used as a bump sensor



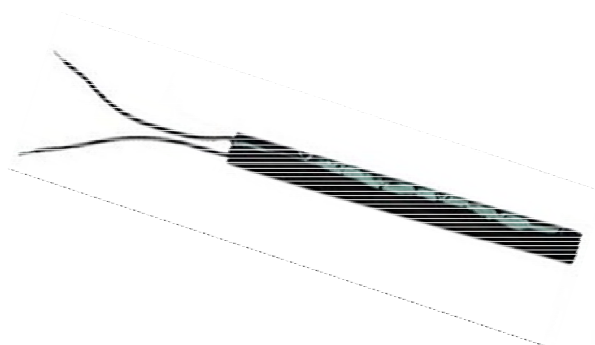
- **Force sensor**
  - A thin layer of force sensing resistors change their resistance in response to applied force
  - Analog output: range of voltages proportional to the applied force
  - Ideal for applications that require some analog precision



Images courtesy of Robot Shop

Contact Sensing

- **Stretch/Bend Sensors**
  - Resistance changes from nominal value when stretched or bent
  - Bi-directional
  - Analog outputs
  - Great for collision avoidance similar to insect antennae or cat whiskers



Images courtesy of Robot Shop

Contact Sensing

- Measurement Models – Roomba bump sensor
  - Simple discrete state measurement model
    - y is either contact or no contact
    - x is either object or no object in the way
    - e.g.,

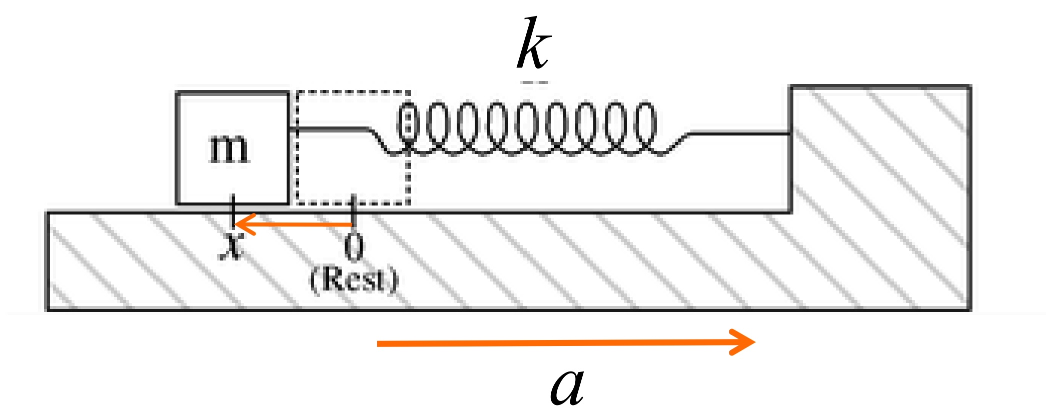
$$p(y_t \mid x_t) = \begin{array}{cc} \begin{bmatrix} 0.99 & 0.1 \\ 0.01 & 0.9 \end{bmatrix} & \begin{array}{l} \text{Contact} \\ \text{No contact} \end{array} \\ \begin{array}{cc} \text{Blocked} & \text{Open} \end{array} & \end{array}$$

OUTLINE

- Measurement Models
- Sensor Characteristics
- Sensors
  - Contact Sensors
  - Inertial Sensors
  - Range Sensors
  - Position Sensors
  - Vision

Inertial Sensors

- Accelerometers
  - Measure acceleration in a single direction by balancing acceleration with spring displacement.

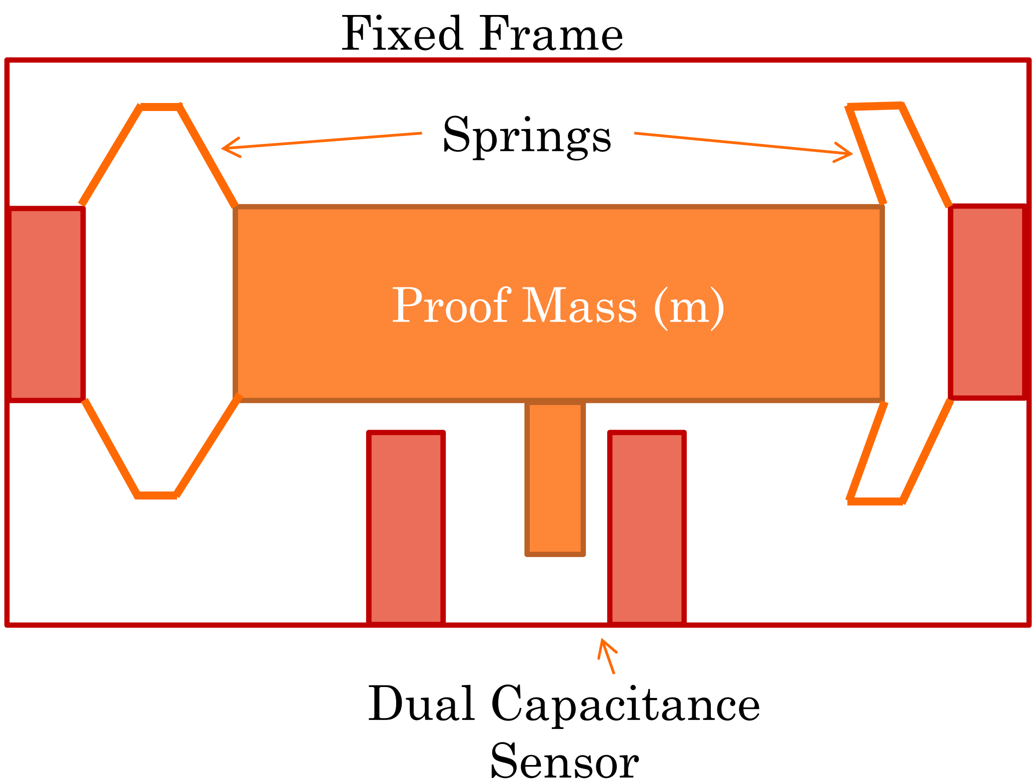


$$F = kx = ma$$

$$a = \frac{kx}{m}$$

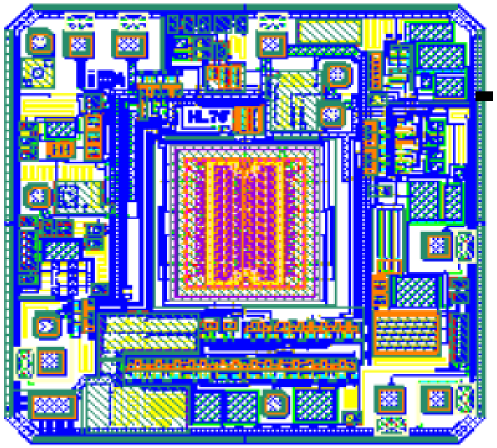
INERTIAL SENSORS

- Analog Devices MEMS Accelerometers
  - iMEMS ADXL150
    - Proof mass suspended from folded springs (0.1 μgm)
    - Capacitive position measurement (10 fF full scale)



Inertial Sensors

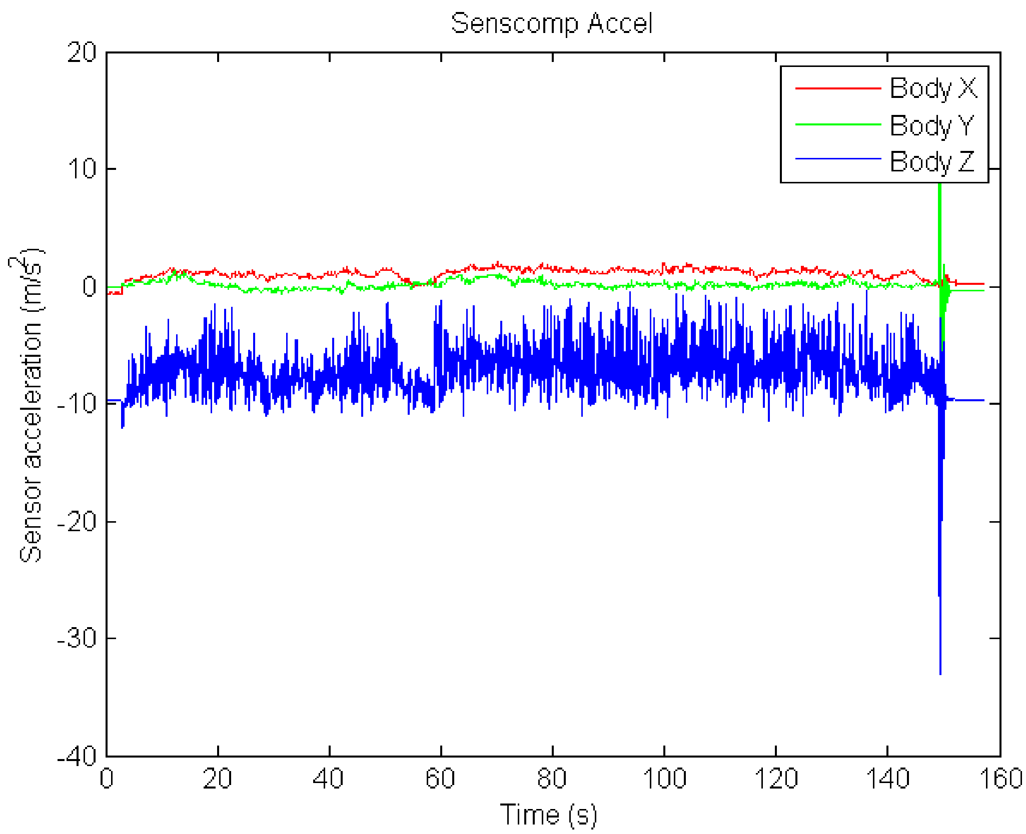
- Accelerometer issues
  - Bias
    - Any bias in measurement causes linear growth in velocity error, quadratic growth in position error
  - Vibration
    - Actual accelerations, cannot be rejected unless known in advance
  - Temperature
    - Changes the spring constant
    - Calibration essential to avoid bias
  - Shock
    - Surprisingly durable
      - 10,000 g shock tolerance



Courtesy of Analog Devices

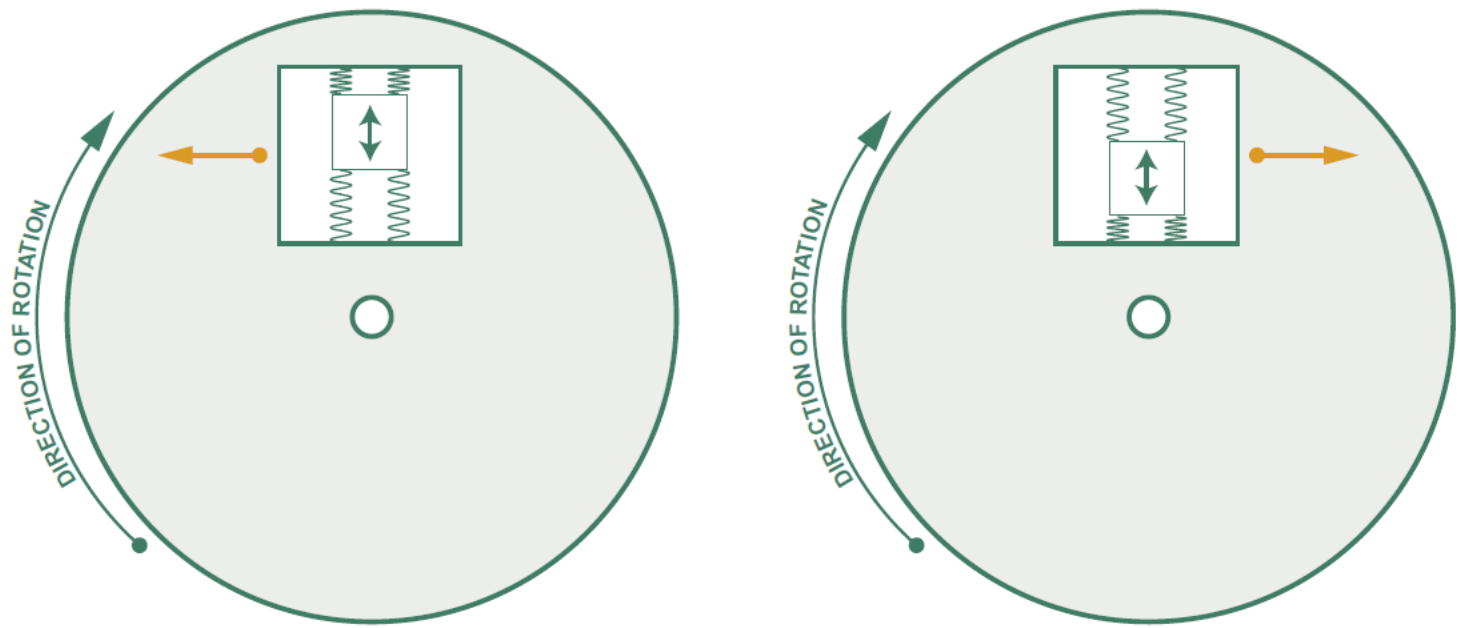
Inertial Sensors

- Accelerometer Data
  - Aeryon Scout quadrotor in hover flight
    - Low pass hardware filter added to raw sensor output
    - High vibration environment, particularly in z axis



INERTIAL SENSORS

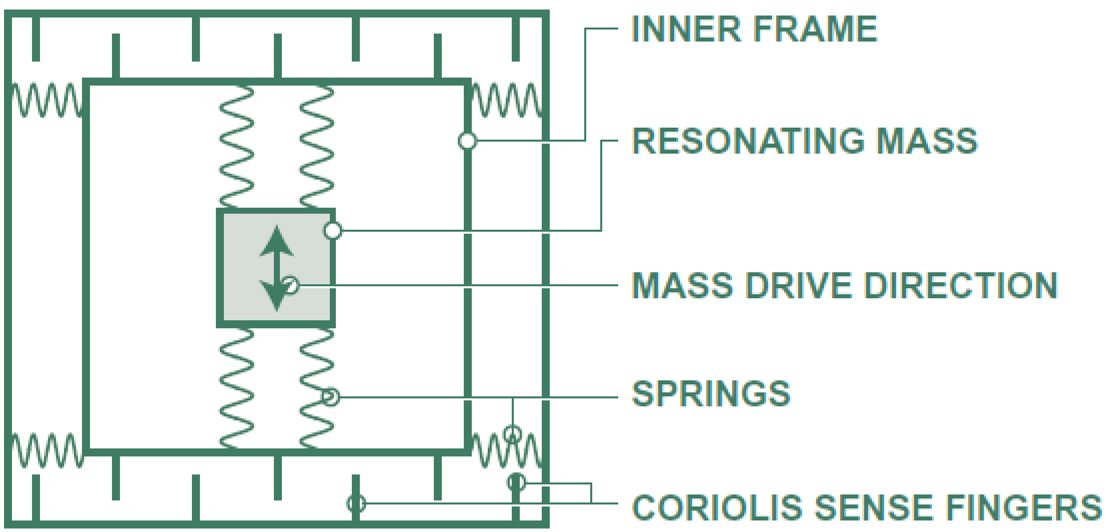
- Gyroscopes
  - Senses Coriolis acceleration: $a_c = 2\omega \times v_{rel}$
  - Vibrating proof mass oscillates in one direction
  - Coriolis acceleration causes perpendicular oscillation at same frequency



Courtesy of Analog Devices

Inertial Sensors

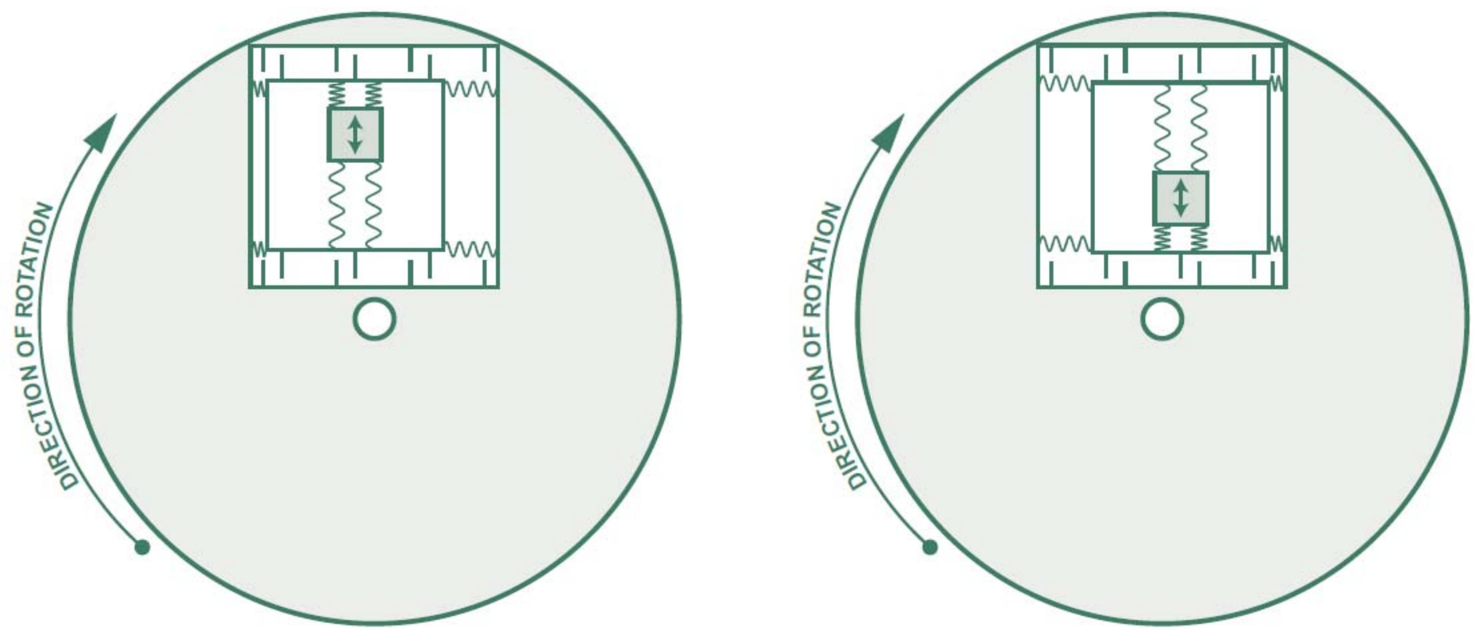
- Gyroscopes
  - Oscillating proof mass structure is suspended by four additional springs perpendicular to axis of oscillation
  - Dual capacitive measurements taken at each corner
  - Measurement signal correlated with oscillation frequency to eliminate other accelerations



Courtesy of Analog Devices

Inertial Sensors

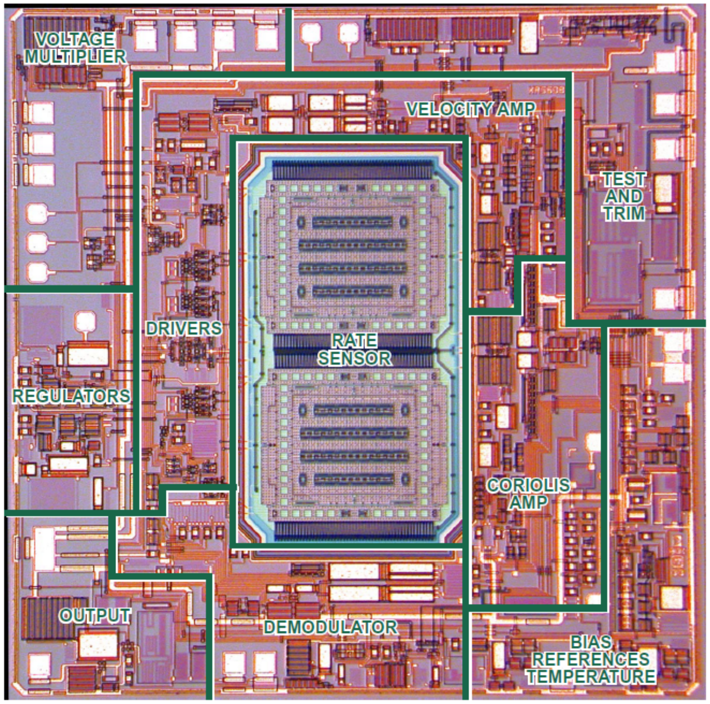
- Gyroscopes
  - As rotation occurs, mass and frame move in predictable pattern



Courtesy of Analog Devices

Inertial Sensors

- Gyroscope measurement issues
  - Vibration
    - Especially at drive frequency
    - Coning particularly troubling
    - Vibration isolation can be used
  - Temperature
    - Affects spring constant
    - Built in compensation
    - Bias estimation
  - Shock
    - Surprisingly durable (10K g)
  - Range
    - Common levels
      - 150, 300, 600, 1200 °/s
    - Turntable @ 33 1/3 rpm
      - = 200 °/s



Courtesy of Analog Devices

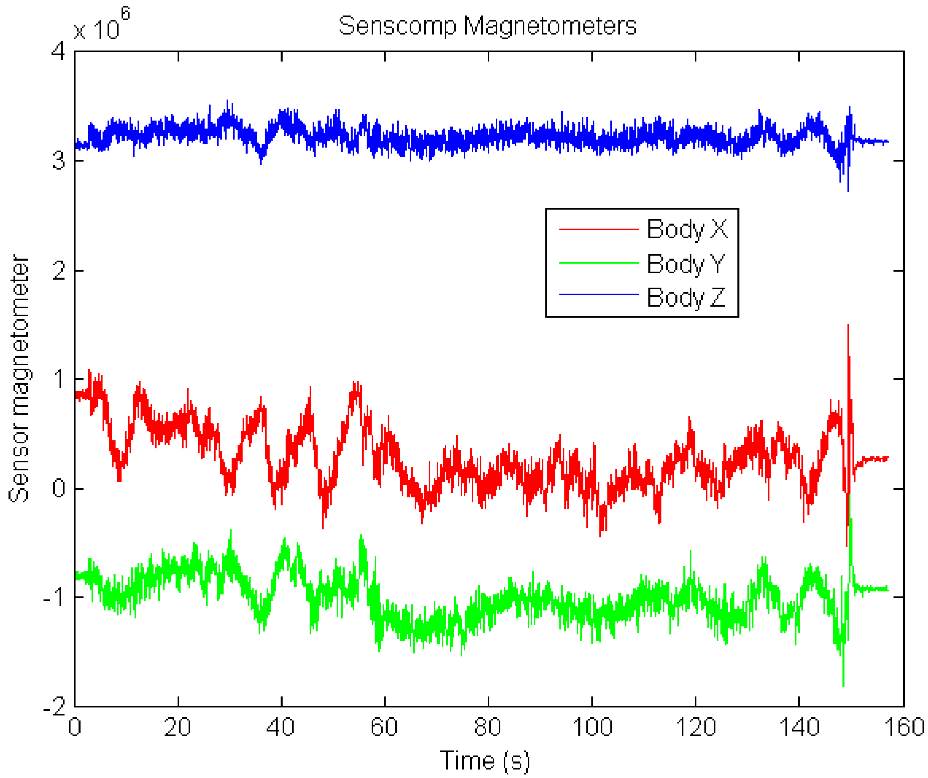
Inertial Sensors

- Magnetometers
  - 4000 year old technology
  - Use earth's magnetic field to provide inertial orientation
  - 3-axis version provides magnetic field vector
  - Good precision, poor accuracy
  - Many disadvantages
    - Earth's magnetic field is weak
    - Field easily disturbed by presence of metal, current, magnets
    - Particularly ineffective indoors (rebar!)

Devantech Compass

Inertial Sensors

- Magnetometer Data
  - Aeryon Scout quadrotor in hover flight
    - Relatively immune to vibration effects
    - Oscillation on similar time scale to vehicle motion
    - Large development effort to account for PCB current



Inertial Sensors

- Measurement models
  - Best categorized as additive Gaussian noise in body coordinates plus a bias term, which is included as an extra state.
    - Noise correlation between axes due to common vibration

$$y_t = Cx_t + \delta_t, \qquad \delta_t \sim N(0, Q_t)$$

    - E.g. for body angular rate, *p*, the gyro measurement model is

$$y_t^p = p_t + \beta_t^p + \delta_t^p,$$

  - Then,

$$p(y_t \mid x_t) = N(y_t - Cx_t, Q_t)$$

    - Here C contains an identity matrix that selects the three body accelerations for the accelerometers, three angular rates for gyros, etc.

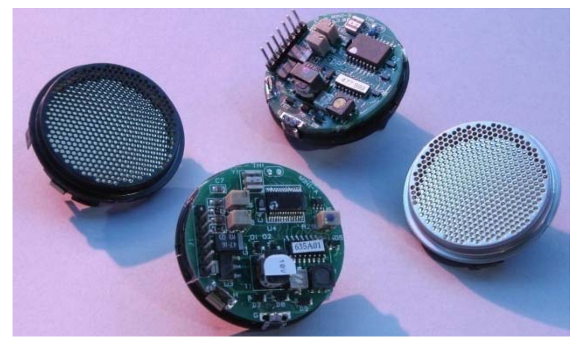
Outline

- Measurement Models
- Sensor Characteristics
- Sensors
  - Contact Sensors
  - Inertial Sensors
  - Range Sensors
  - Position Sensors
  - Vision

RANGE SENSORS

- **Sonic Range Finder (SODAR, Sonic Ranger)**
- Emits a sonic pulse and measures the time of flight of the first or multiple echoed pulse(s)
  - Devantech SRF08: 40kHz
  - Senscomp Mini-AE: 50 kHz
- Pulse must be followed by a blanking period for return reception
  - Collocated speaker and microphone
- Very effective in underwater applications
- Speed of sound
  - 343 m/s in air
  - 1497 m/s in freshwater

SRF08



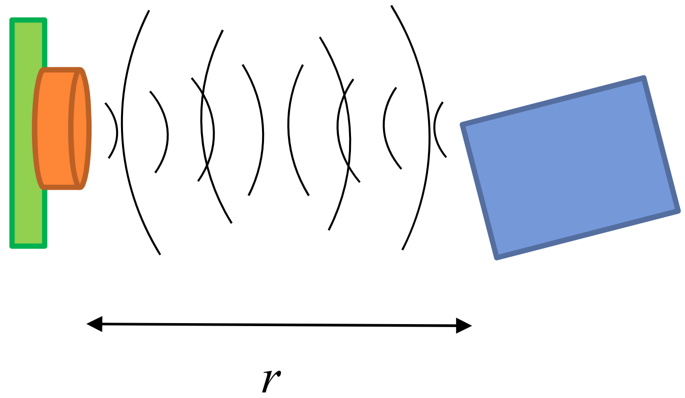
Mini-AE

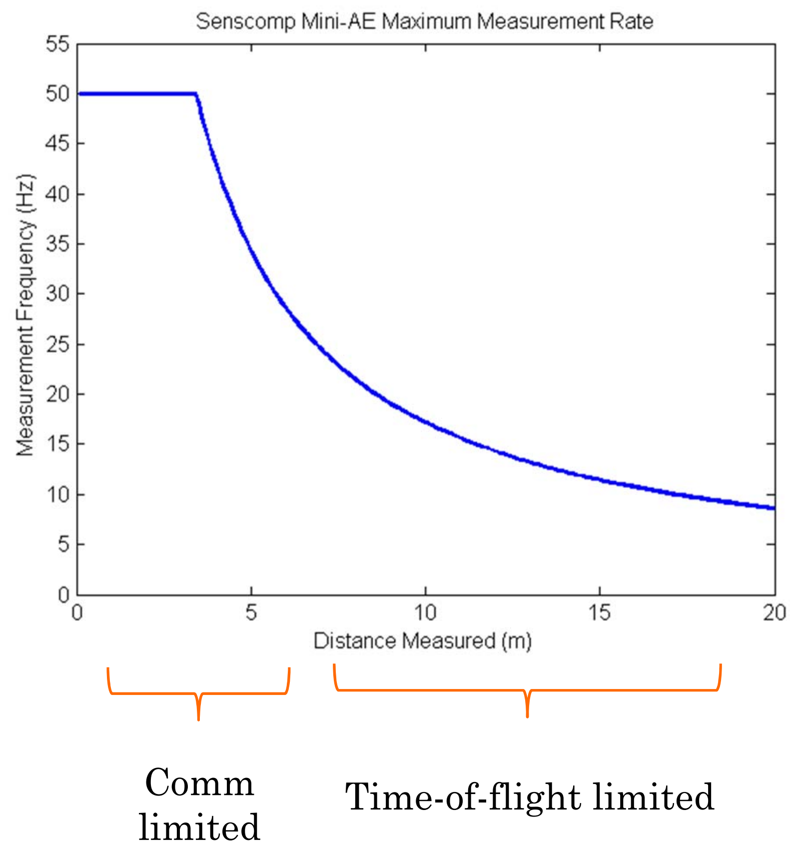
Senix TSPC-21S-232

Range Sensors

- **SODAR**
  - Must wait for return signal
  - Minimum distance defined by blanking time
  - Maximum distance defines update rate, limited by dissipation of pulse energy

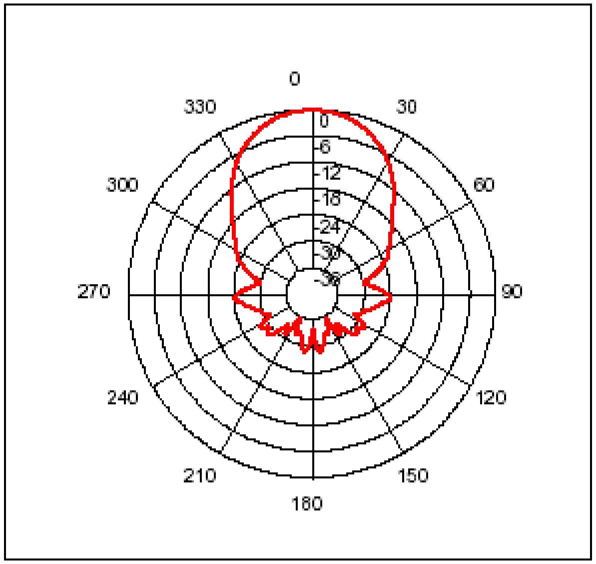
    - SRF08: Max 12Hz
    - Mini-AE: Max 50 Hz



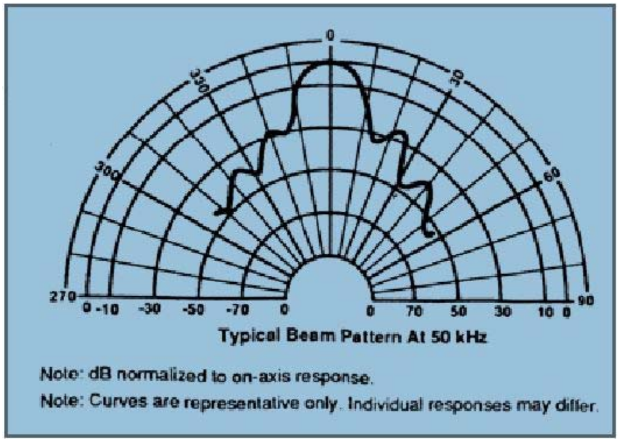


Range Sensors

- **SODAR**
- Limited operating range
  - Devantech SRF08: 10 cm to 2 m
  - Senscomp Mini-AE: 15cm to 12m
- Sound signal is hard to direct
  - Wide cone ~ 20degrees
- Must be careful with multiple SODAR
  - Angle in distinct directions
  - A real challenge in multi-robot applications



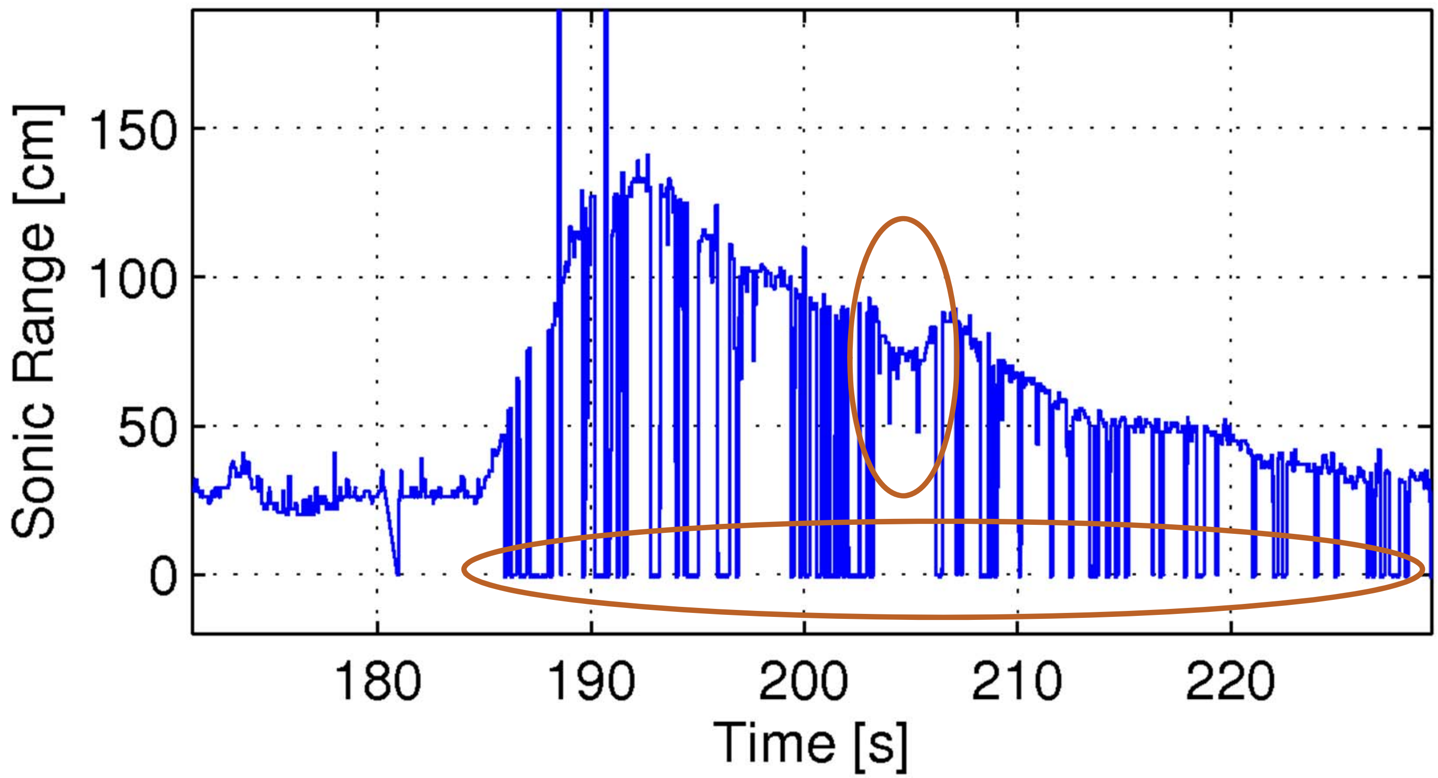
SRF08



Mini-AE

RANGE SENSORS

- **SODAR**
  - Data suffers from false echoes, missed readings



  - Quadrotor flight data from SRF08 (STARMAC 2005)

Range Sensors

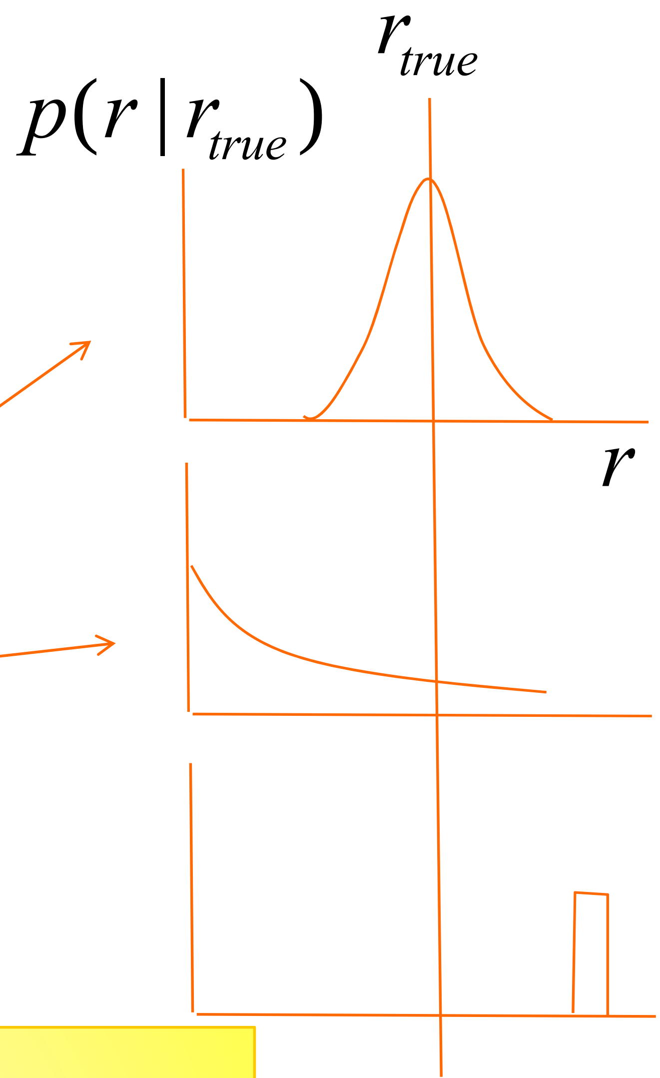
- Sonar Measurement Model
  - Get range to nearest object in field of view

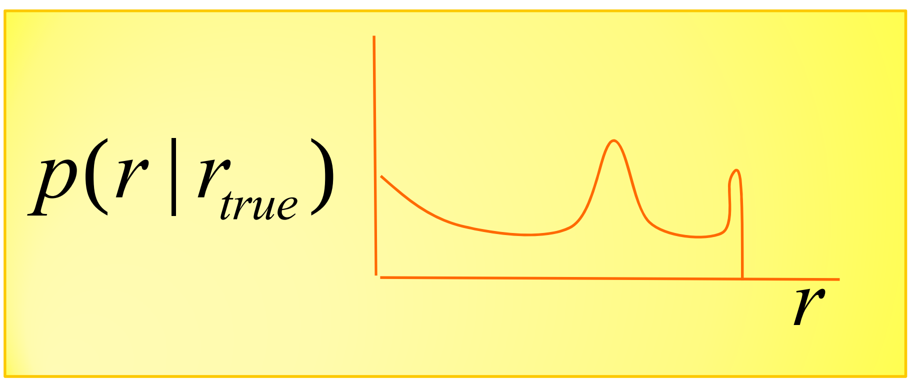
$$y^r = \sqrt{\left(m_x - x_t\right)^2 + \left(m_y - y_t\right)^2}$$

    - Where m is the position of the closest object
      - Raises the questions, which is the closest object!?
  - Noise on this measurement comes in many forms
    - Accuracy of measurement results in Gaussian about true range due to time of flight measurement
    - False echoes result in potentially completely incorrect measurements
    - Missed echoes result in max range

Range Sensors

- Sonar Measurement Model
  - Noise profile
    - Gaussian about true range
    - False echoes, exponential in range measurement
    - Max range
  - Full model





Range Sensors

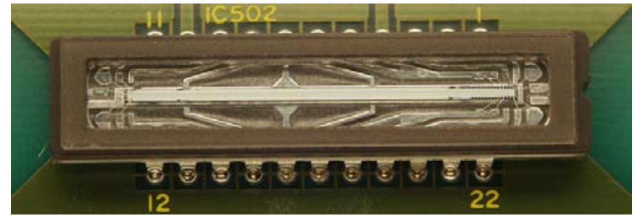


Image courtesy of Wikipedia

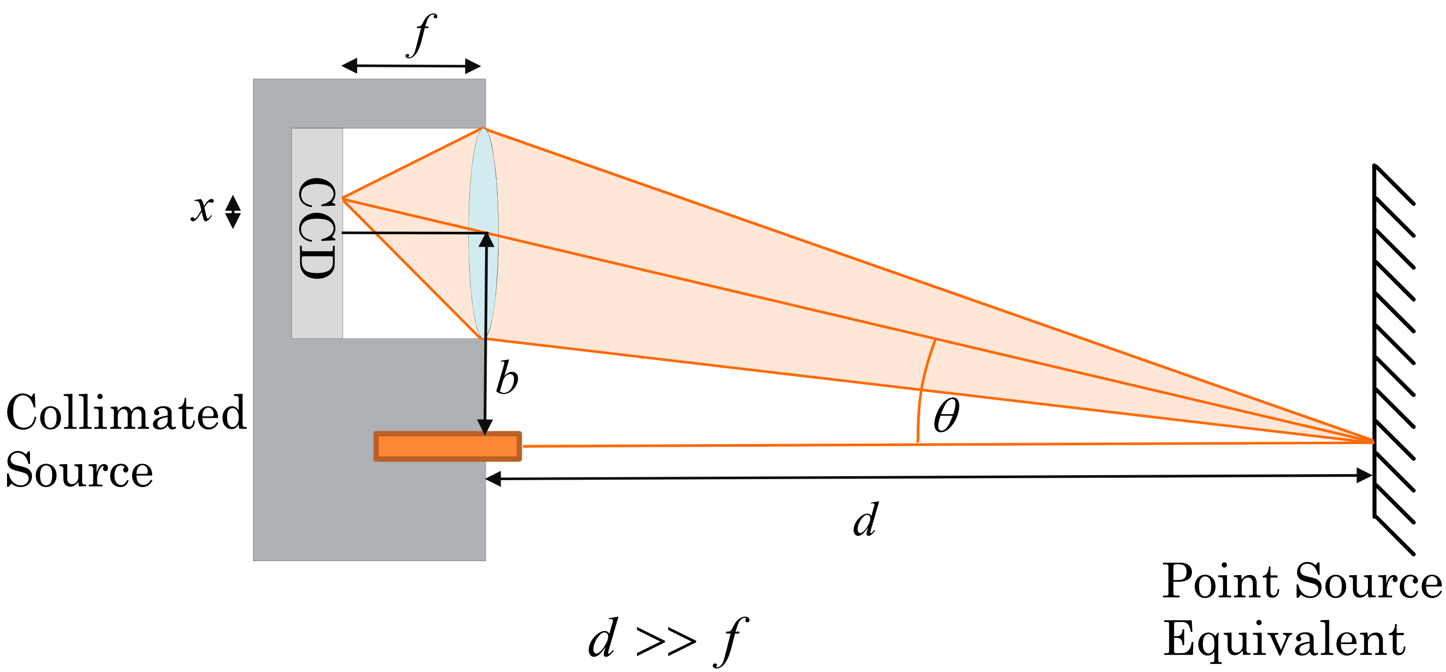
- **Infrared Range Finder (Ranger)**
- Emits an infrared light beam and measures the angle of the reflected beam to triangulate the distance to a surface
- Limited operating range, typically 30 cm to 1.5 m
- Reflected light is captured by an image sensor
  - Array of photoelectric light sensors used to capture image data
- Infrared is the wavelength of choice to minimize "noise" from ambient light
  - Modulate emitted light at a fixed frequency
- Common model: Sharp GP2D12 sensor

Images courtesy of Robot Shop

RANGE SENSORS

- IR Ranger
  - Triangulation

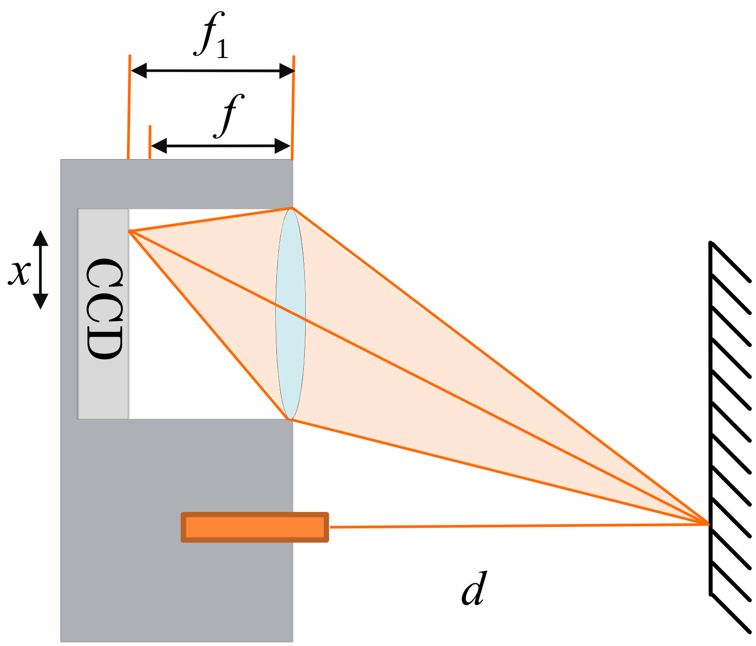
$$\tan\theta = \frac{b}{d} \qquad \tan\theta = \frac{x}{f} \qquad d = \frac{fb}{x}$$



$$d >> f$$

Range Sensors

- IR Ranger
  - As the surface moves closer, the dot on the CCD moves up, and the focal plane moves away
  - Negligible for small focal lengths



$$f_1 = \left(\frac{1}{f} - \frac{1}{d}\right)^{-1}$$

If f = 0.01m, d = 1m

$$= \left(\frac{1}{0.01} - \frac{1}{1}\right)^{-1} = 0.0101m$$

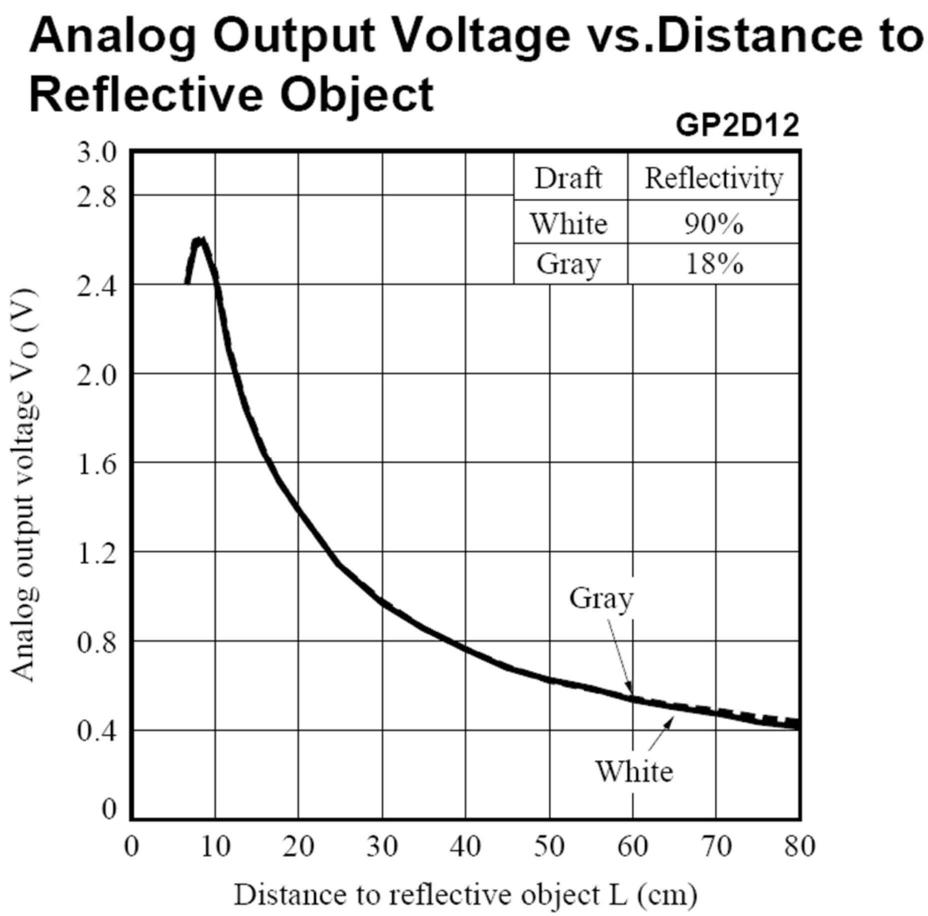
If f = 0.01m, d = 0.2m

$$= \left(\frac{1}{0.01} - \frac{1}{0.2}\right)^{-1} = 0.0105m$$

Range Sensors

- Typical IR Ranger response
  - Measures angle based on illuminated CCD position
  - 1/x decay clearly visible

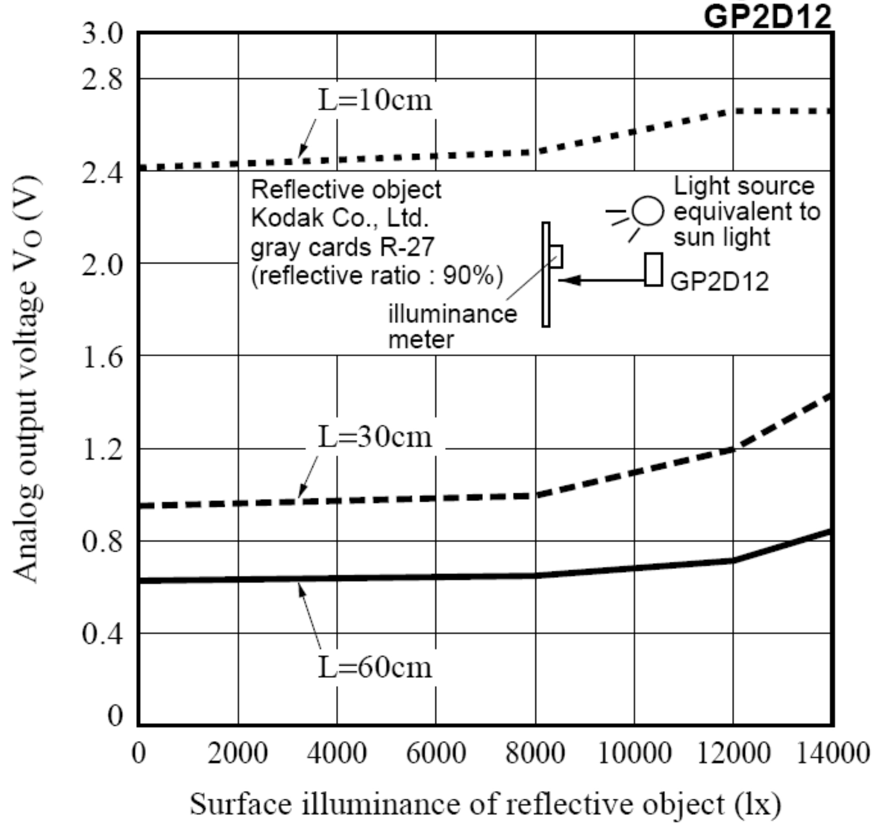
$$d = \frac{fb}{x}$$



RANGE SENSORS

- IR Ranger
- What to do about ambient infrared waves?
  - Modulate the emitted light with a 40 kHz carrier wave
  - Filter the return response to the imager by selecting only that frequency
- Surface illuminance affects sensor
  - Brightly lit surfaces skew reading
- The Sharp GP2D12 sensor stays relatively insensitive to surface illuminance up to 8000 lux
- 500 lux is the level of a typical business office
- 25,000 lux is the daylight level
- The GP2D12 is better suited to indoor use

Analog Output Voltage vs. Surface Illuminance of Reflective Object



RANGE SENSORS

- IR Measurement model
  - Noise comes from measurement of pixel location, x

$$d = \frac{fb}{x + \delta} \qquad \delta \sim N(0, Q)$$

    - For large *x*/small *d*, errors in *x* are small relative to magnitude of *x*, limited effect
    - For small *x*/large *d*, errors in *x* are large relative to magnitude of *x*, major effect

Range Sensors

- IR Measurement model
  - Range to point directly in front of sensor

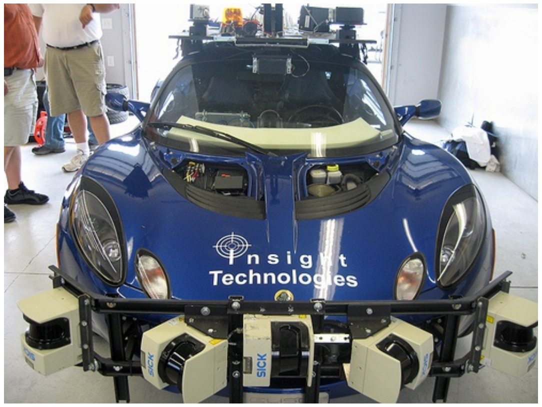
$$d = \sqrt{\left(m_x - x_t\right)^2 + \left(m_y - y_t\right)^2}$$

  - Rearranging to form measurement model

$$y^r = x + \delta = \frac{fb}{d} + \delta = \frac{fb}{\sqrt{\left(m_x - x_t\right)^2 + \left(m_y - y_t\right)^2}} + \delta$$

Range Sensors

- Laser Scanners
  - Time of flight sensors, improved accuracy, frequency
  - Significant additional complexity, weight
  - Robotics Industry Standard: SICK Scanners
  - Robotics Industry Platinum Edition: Reigl, Velodyne
  - Lightweight Entries: Hokuyo



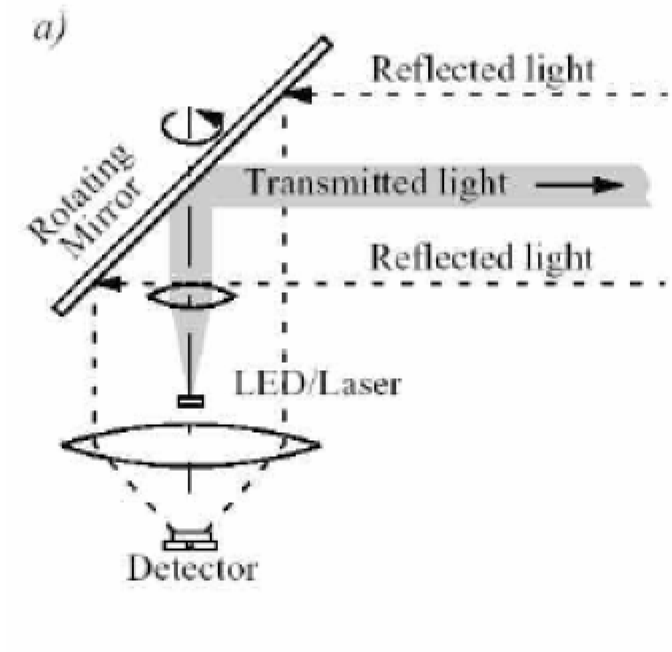
SICK 240 degree, 20 Hz, 30m

Velodyne 360 x 180 degree, 10 Hz 50m

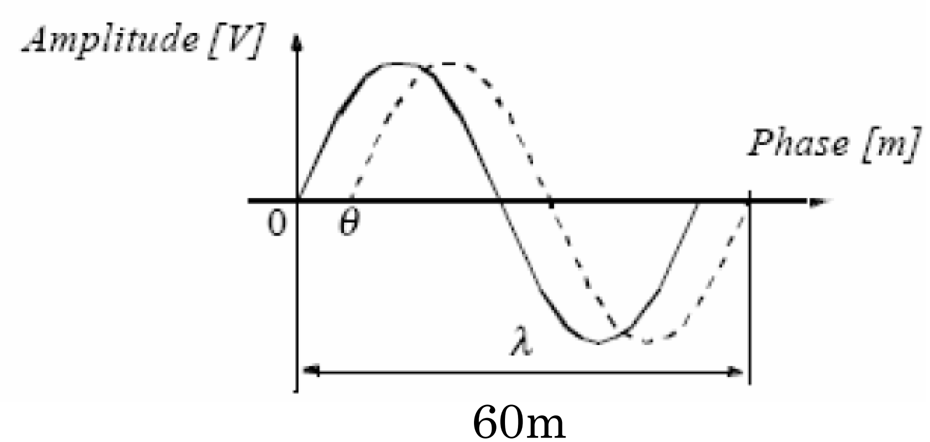
Hokuyo URG,10 Hz, 4m, 160gm

Range Sensors

- Laser Scanners
  - Revolving mirror creates circular beam
  - Emitted light is amplitude modulated
    - e.g. at 5 MHz, 60m wavelength
  - Transmitted and reflected light phase shift compared

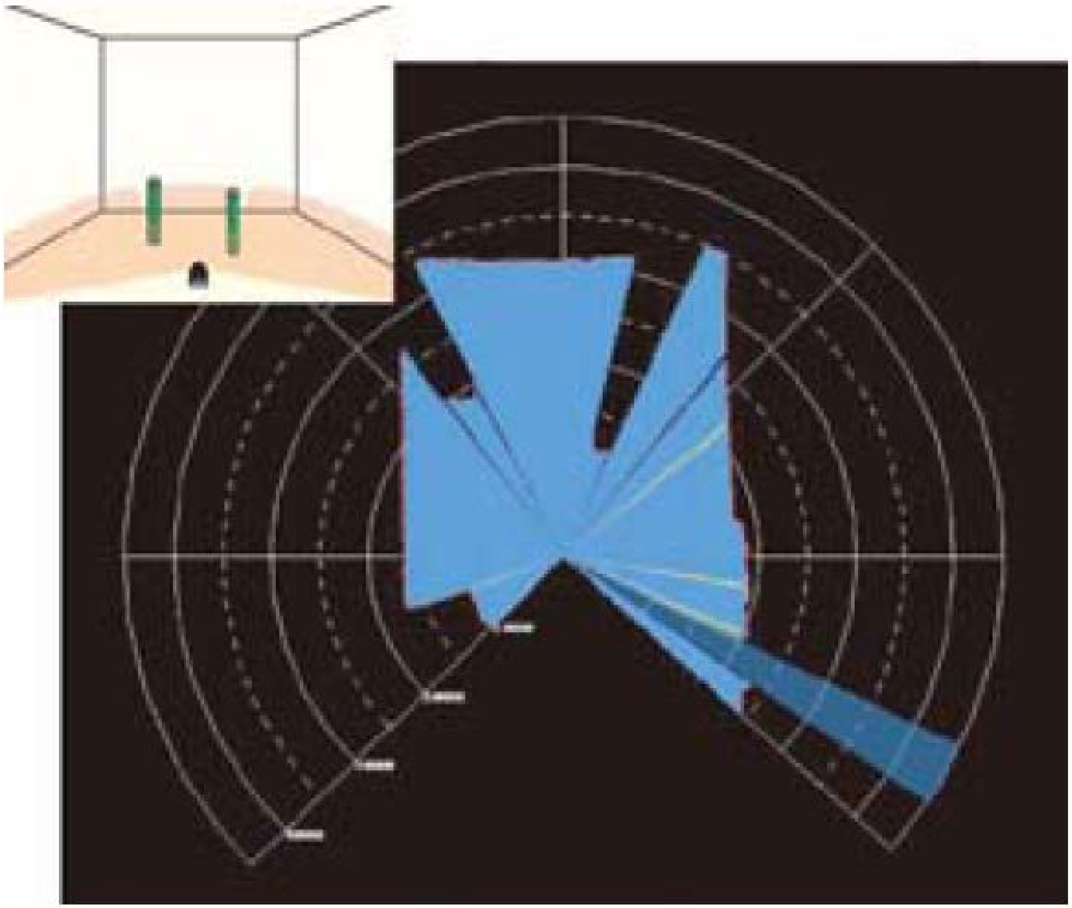


$$d = \frac{\theta\lambda}{4\pi}$$



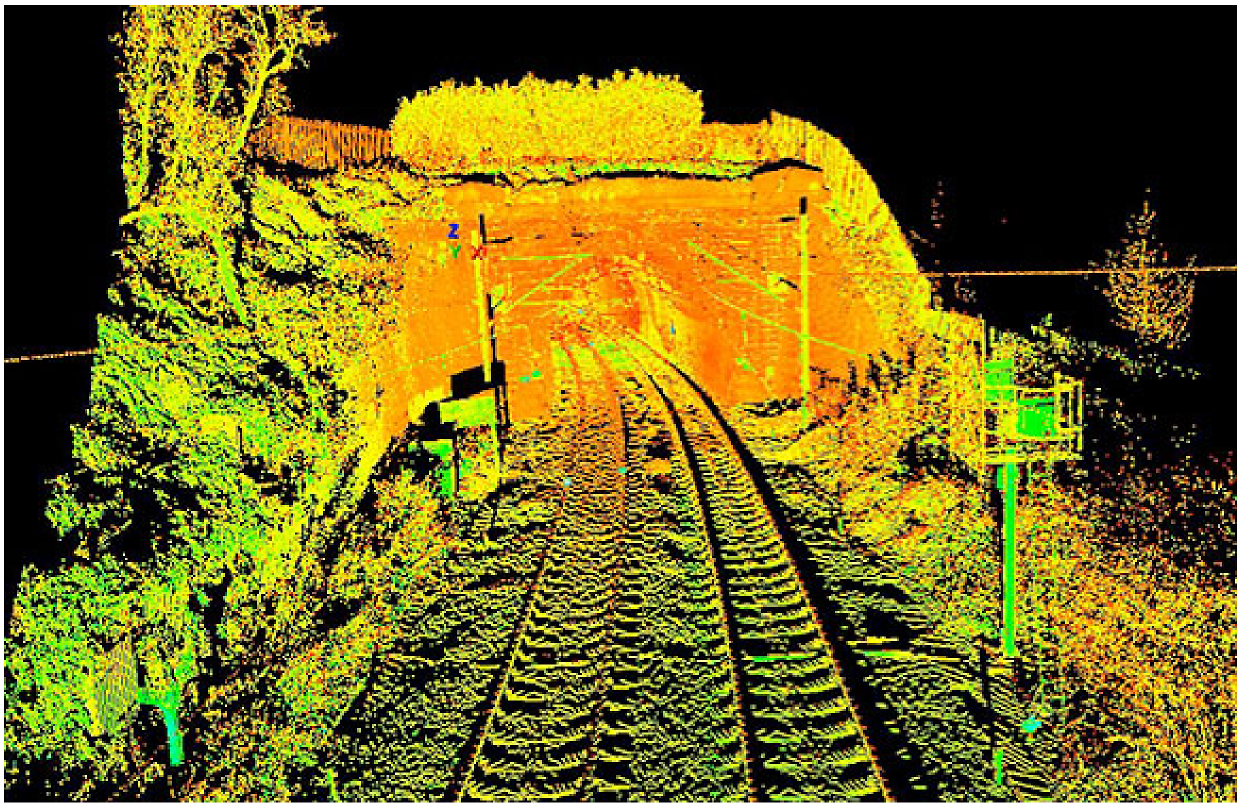
Range Sensors

- Laser Scanners
  - Hokuyo output for a single scan in a room with two posts



Range Sensors

- Laser Scanners
  - Reigl 3D scan, surface normal colouration



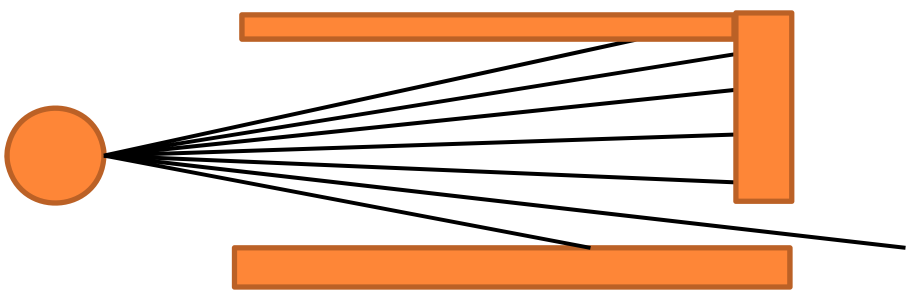
Range Sensors

- Laser Scanner Measurement Model
  - Returns a range to the closest objects at a set of bearings relative to the vehicle heading
    - Scanner bearings

$$\phi^s = \begin{bmatrix} -\phi^s_{\max} & \dots & \phi^s_{\max} \end{bmatrix}$$

    - Scanner ranges

$$r^s = \begin{bmatrix} r^s_1 & \dots & r^s_J \end{bmatrix} \qquad r^s_j \in \left[0, r^s_{\max}\right]$$



Range Sensors

- Laser Scanner Measurement Model
  - Get range and bearing to each object

$$r_i^s = \sqrt{\left(m_x^i - x_{1,t}\right)^2 + \left(m_y^i - x_{2,t}\right)^2}$$

$$\phi_i^s = \tan^{-1}\left(\frac{m_y^i - x_{2,t}}{m_x^i - x_{1,t}}\right) - x_{3,t}$$

  - Noise sources similar to IR ranger

Outline

- Measurement Models
  - Definition
  - Block Diagrams
  - LTI System Model
- Sensors Characteristics
- Sensor Examples
  - Contact Sensors
  - Inertial Sensors
  - Range Sensors
  - Position Sensors
  - Vision

POSITION SENSORS

- Encoders
  - Very useful for robotic arms, CNC machining
  - Quite useful for wheeled vehicles
    - When used for velocity and position estimation, must consider possibility of wheel slip
    - Usually higher frequency output than GPS, can be used to augment state estimation

Courtesy of Olympus

Courtesy of US Digital

POSITION SENSORS

- **Incremental Encoder**

- Generates a pulse for a given increment of shaft rotation
- Given the resolution of the encoder and dimensions of the wheel that it is attached to:
  - Rotary displacement can be determined by counting the number of output pulses
  - Rotational velocity can be determined by timing the rising edges of the output pulses
- Resolution of the encoder depends on the number of slits

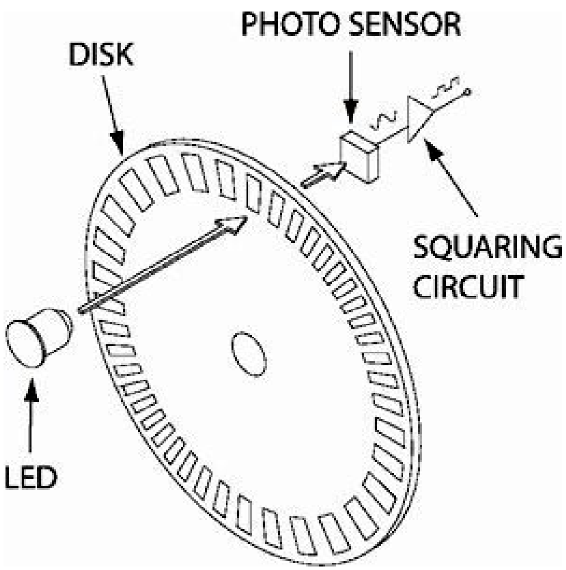


Image courtesy of Encoder Products Company

POSITION SENSORS

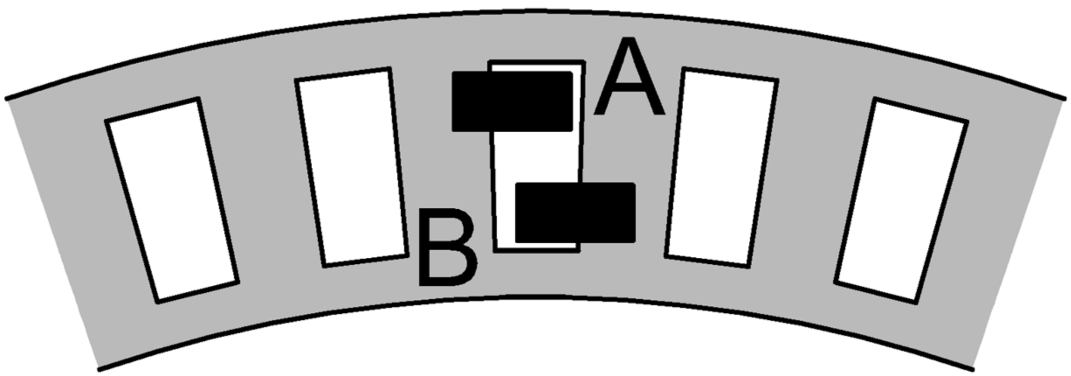
- **Incremental Encoder - Quadrature**

- Rotational direction can be determined using two sensors placed exactly one half slit width apart

- As the disc rotates, light shining through each slit produce two output signals that are 90° out of phase

- If the disc rotates counter-clockwise, sensor B will always be activated first

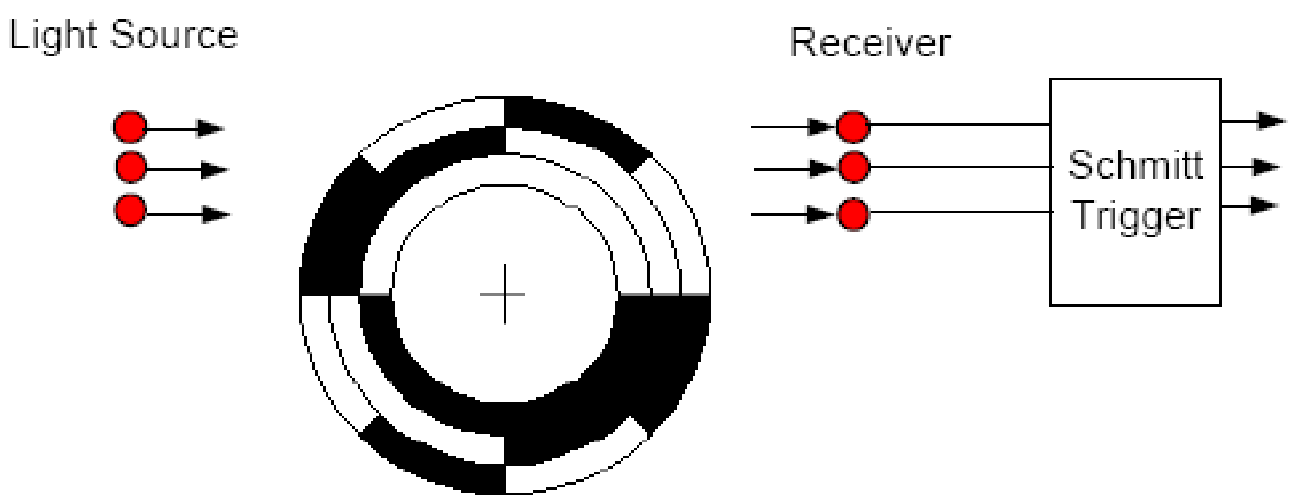
- If the disc rotates clockwise, sensor A will always be activated first



Position Sensors

- **Absolute Encoder**

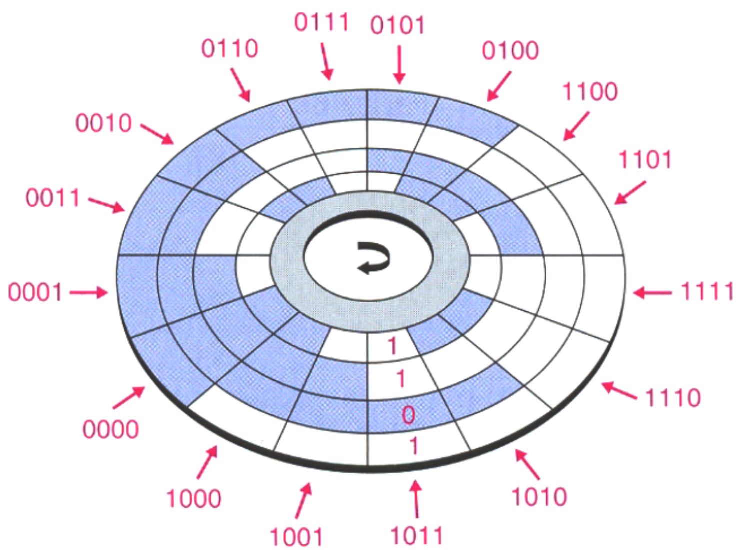
- Consists of a circular clear disc imprinted with rows of broken concentric arcs of opaque material
- A light source and sensor is assigned to each row
- The arcs are arranged such that the pattern of sensor activation is a unique encoding of the position of the shaft



Position Sensors

- **Absolute Encoder**

- The output from each light source/sensor pair represents one bit
- A 4 bit absolute encoder is shown below
- If light passes through, the sensor outputs a logic 1, 0 otherwise
- The series of binary numbers encodes the shaft position – Gray code



Rotation range (deg.)	Gray code
0-22.5	0000
22.5-45	0001
45-67.5	0011
67.5-90	0010
90-112.5	0110
112.5-135	0111
135-157.5	0101
15.75-180	0100
180-202.5	1100
202.5-225	1101
225-247.5	1111
247.5-270	1110
270-292.5	1010
292.5-315	1011
315-337.5	1001
337.5-360	1000

Position Sensors

- Encoder Measurement Model
  - The encoder returns wheel position

$$y_t = \phi_t$$

    - Absolute encoder outputs directly
    - Relative encoder requires a pulse counter, can miss pulses
    - Converted to a distance traveled by wheel diameter
    - Must be related to the vehicle state
      - For two wheel robot, *x, y, θ, v, w*

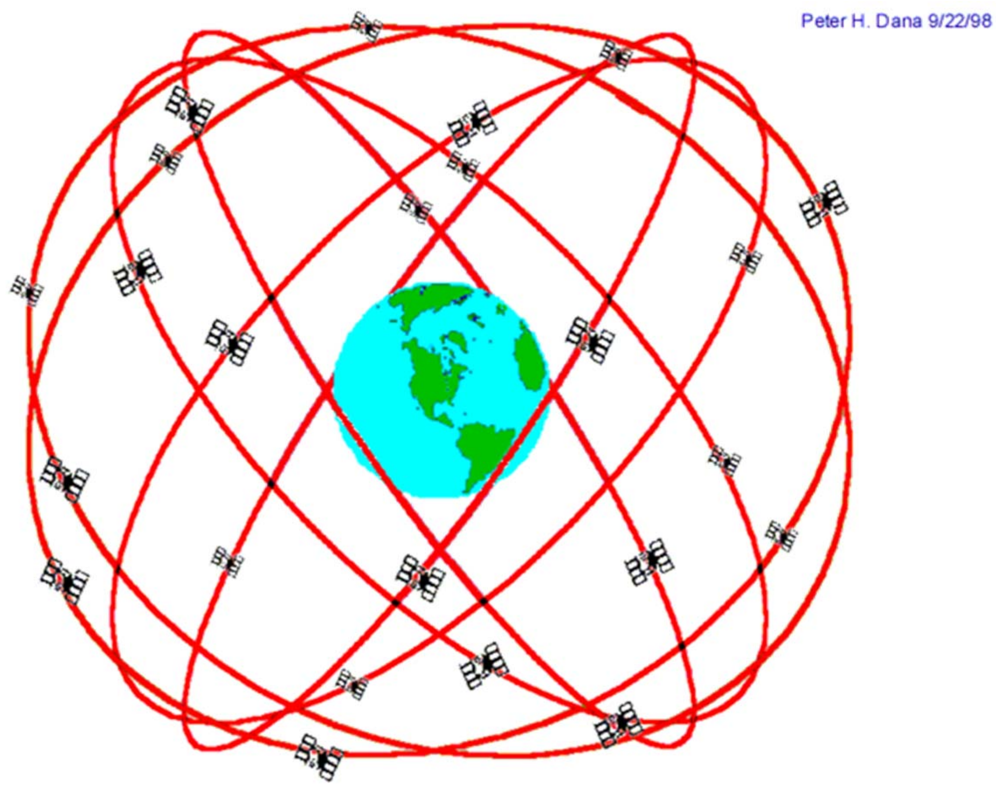
$$y_t = \frac{v_t dt}{r}$$

  - Errors are definitely not Gaussian
    - Wheel slip
    - Quantization due to pulse counts per interval
    - Much worse accuracy at low speeds

Position Sensors

- **Global Positioning System (GPS)**

- Position determined based on time of flight
  - Need 4 geometrically distributed sources
  - Need 1 nanosecond timing for sub-meter position resolution



GPS Nominal Constellation
24 Satellites in 6 Orbital Planes
4 Satellites in each Plane
20,200 km Altitudes, 55 Degree Inclination

12 hour period

POSITION SENSORS

- **How can GPS possibly work?!**

- Minimal power
  - Satellites produce ~25 Watt signals
  - Receivers see -160 dBW signal (10^{-16} Watts)

- Precise Satellite position information
  - V_{sv} = 3.9 km/s in ECEF

- Precise timing
  - 1 μsec = 300 m range error
  - Satellite relativistic effect = 38 μsec/day or 12 km error!
    - +45 μsec/day General Relativity: clocks faster in lower gravity
    - -7 μsec/day Special Relativity: clocks slower at higher velocity

- Consistent earth model
  - Rotation axis varies
  - Elliptical shape approximation
  - Gravity not constant at mean sea level or normal to surface

Position Sensors

- **But GPS does work after all!!**

- Can manage on minimal power
  - Gold Codes allow tracking of extremely weak signals

- Precise satellite positions available
  - Control Segment estimates orbital parameters to 2 m position accuracy

- Precise timing feasible
  - Satellites use rubidium and cesium clocks (drift 10^{-9} s/day)

- Thanks to Einstein for figuring out relativity

- WGS84 earth model
  - WGS84 standard defines ECEF coordinates, earth shape and vertical direction

POSITION SENSORS

- GPS Hardware

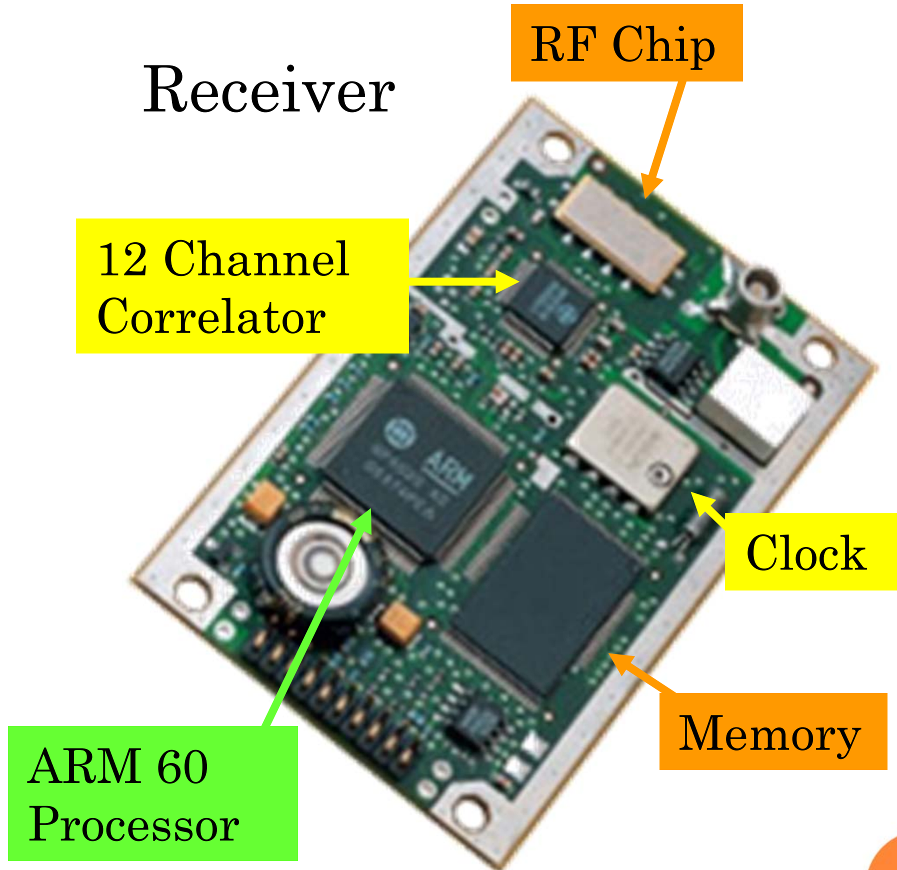
Space Vehicle (SV)

GPS Block II-F satellite

Control Segment (CS)

Satellite-tracking-station on Hawaii

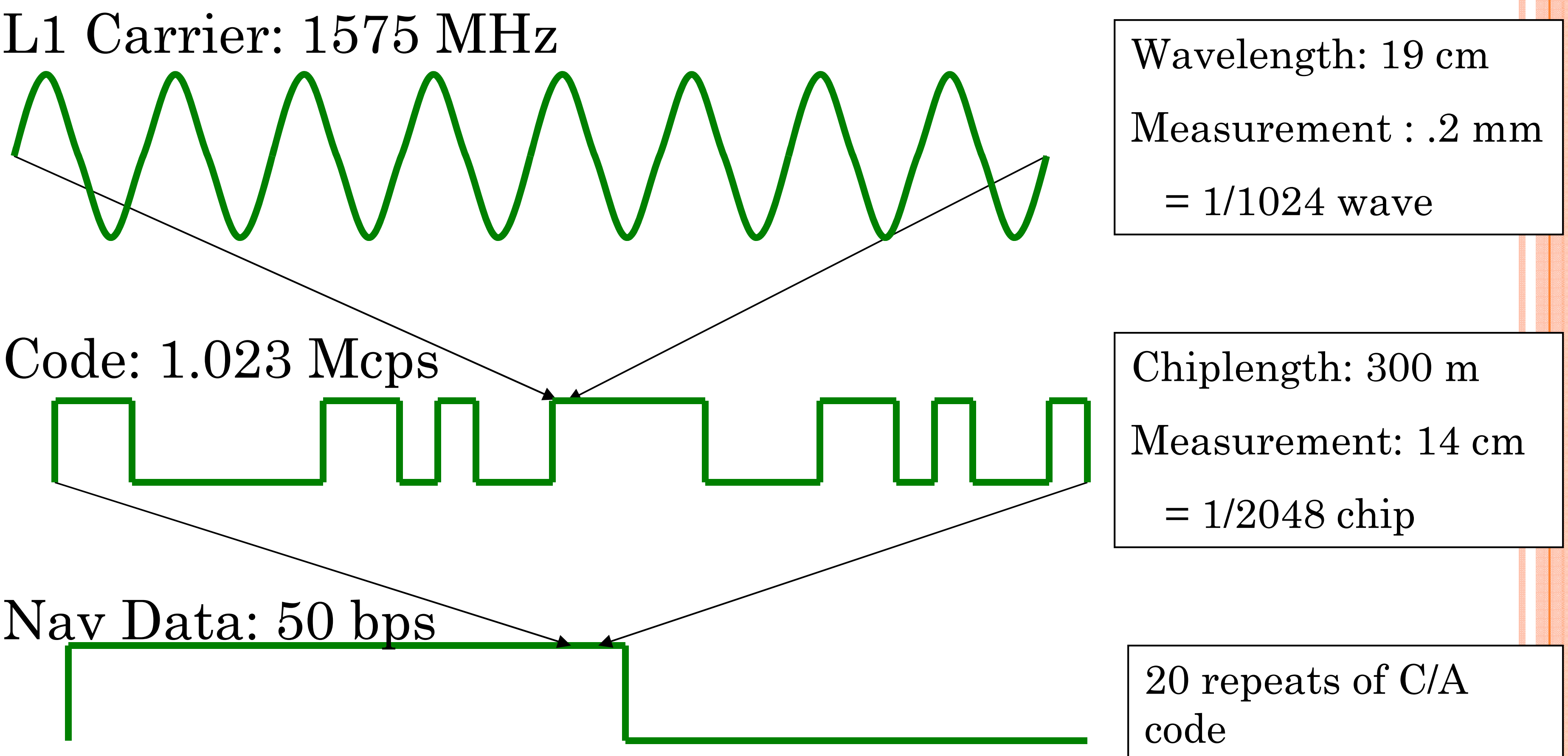
Receiver



Novatel Superstar II

POSITION SENSORS

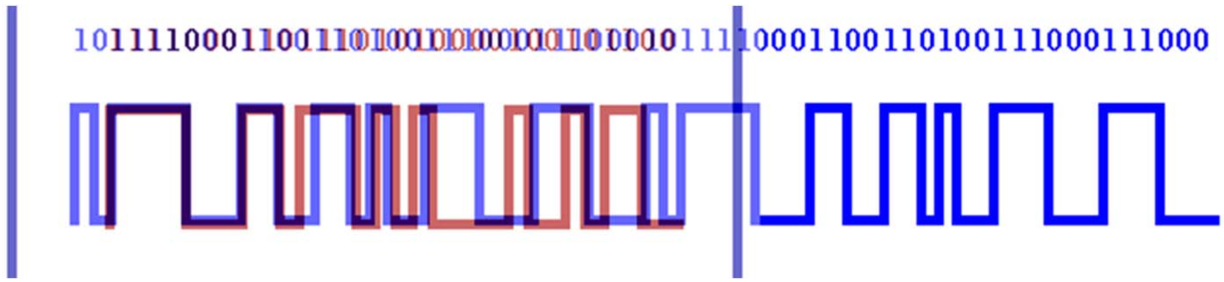
- GPS Satellite Signals
  - Travel at light speed, *c*



Position Sensors

- GPS Signal correlation
  - 32 unique 1024 chip gold codes which are orthogonal

- None



- Half

- Full

Position Sensors

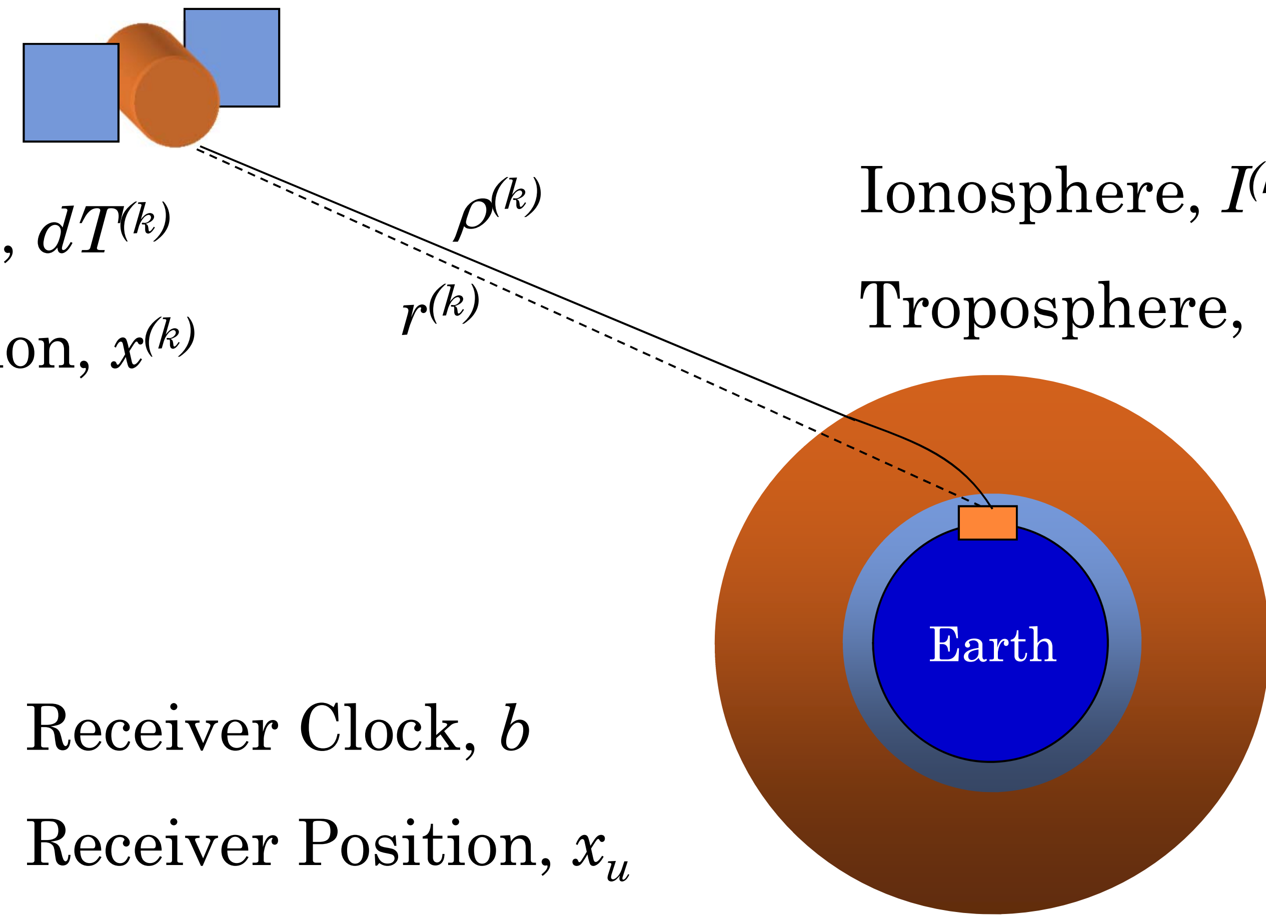
GPS Measurements & Error Sources

SV Clock, $dT^{(k)}$

SV Position, $x^{(k)}$

Ionosphere, $I^{(k)}$

Troposphere, $T^{(k)}$



Receiver Clock, b

Receiver Position, x_u

Receiver Measurement and Multi-path, $\varepsilon_\rho^{(k)}$

Position Sensors

- Code Phase Position Fix Measurement Model
  - Nonlinear measurement model of pseudorange, $\rho^{(k)}$ in meters

Range Clocks Atmospheric Other

$$\rho^{(k)} = ||x^{(k)} - x_u|| + b - c(dT^{(k)}) + I^{(k)} + T^{(k)} + \epsilon_\rho^{(k)}$$

- Nonlinear least squares (NLLS) estimation used
  - Combine pseudorange measurements to form an estimate of receiver position and clock bias
- Can be augmented with motion model if known
  - Many GPS receivers include options for walking, car, aircraft etc.

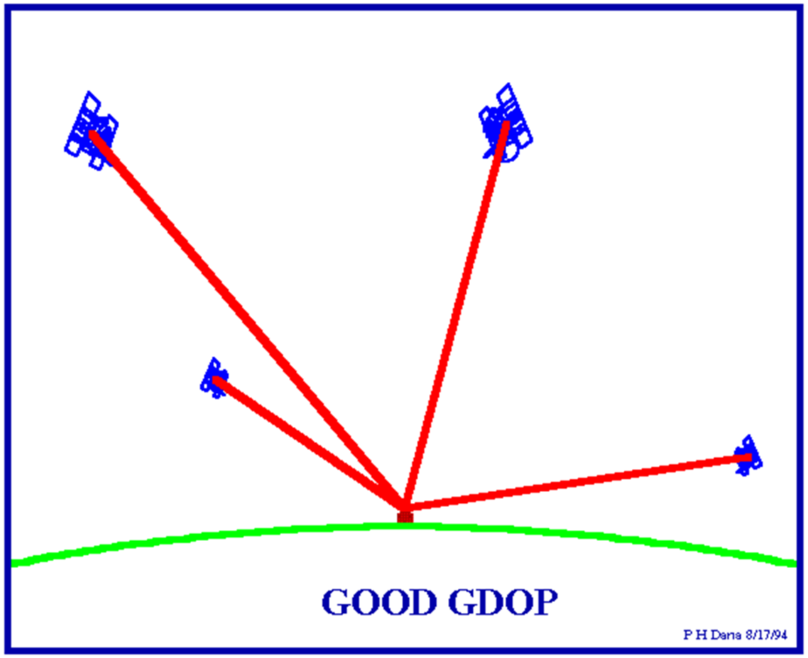
POSITION SENSORS

- Main GPS Error Sources
  - Geometry, independence of pseudorange measurements
  - Atmospheric delays
  - Multipath Signal Interference
- Other contributing factors
  - Satellite Position Calculation
  - Satellite Clock Corrections
  - Code Tracking Error

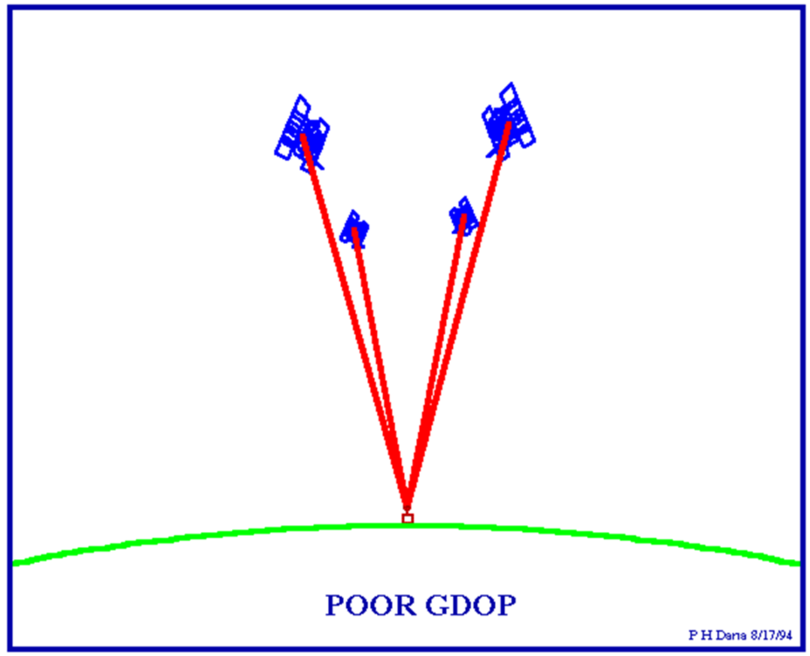
- Errors often cited in Circular Error Probable
  - 5m 50% CEP indicates that 50% of measurements will lie within a 5m circle about the average

POSITION SENSORS

- Geometric Dilution of Precision (GDOP)
  - The lower the better
    - 1 – ideal, above 8 becomes difficult to use
  - Also available PDOP, HDOP,VDOP, TDOP



GDOP = 1.5



GDOP = 6

Position Sensors

- GDOP is calculated from unit vectors to SVs
- Clock bias is added as fourth column
- Derived from errors in least squares calculation

$$A = \begin{bmatrix} \frac{x^1 - x_u}{r^1} & \frac{y^1 - y_u}{r^1} & \frac{z^1 - z_u}{r^1} & 1 \\ \frac{x^2 - x_u}{r^2} & \frac{y^2 - y_u}{r^2} & \frac{z^2 - z_u}{r^2} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{x^n - x_u}{r^n} & \frac{y^n - y_u}{r^n} & \frac{z^n - z_u}{r^n} & 1 \end{bmatrix}$$

$$Q = \left(A^T A\right)^{-1}$$

$$Q = \begin{bmatrix} EDOP^2 & \bullet & \bullet & \bullet \\ \bullet & NDOP^2 & \bullet & \bullet \\ \bullet & \bullet & VDOP^2 & \bullet \\ \bullet & \bullet & \bullet & TDOP^2 \end{bmatrix}$$

$$GDOP = \sqrt{EDOP^2 + NDOP^2 + VDOP^2 + TDOP^2}$$

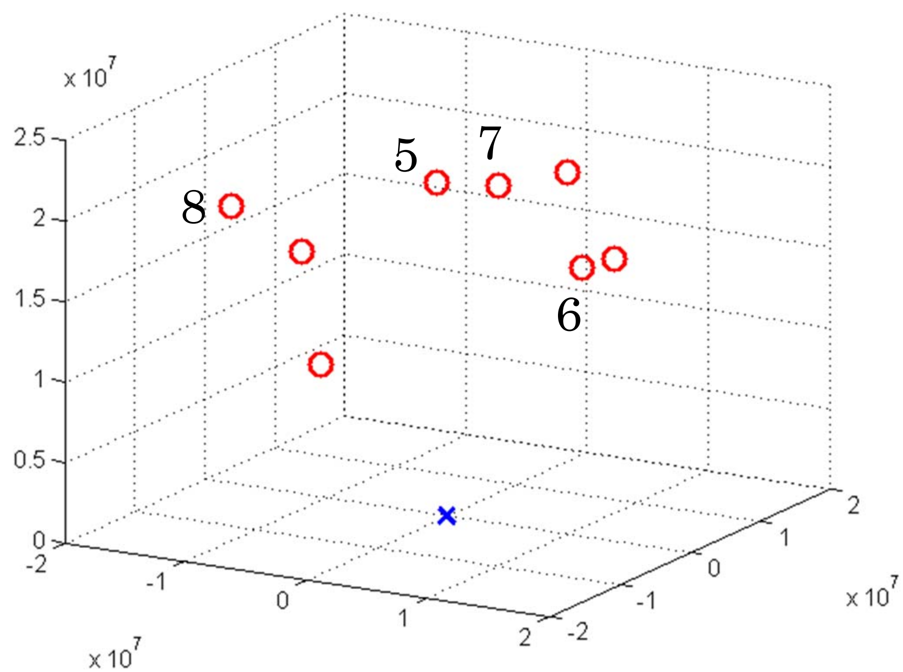
$$PDOP = \sqrt{EDOP^2 + NDOP^2 + VDOP^2}$$

$$HDOP = \sqrt{EDOP^2 + NDOP^2}$$

Position Sensors

- GDOP Examples

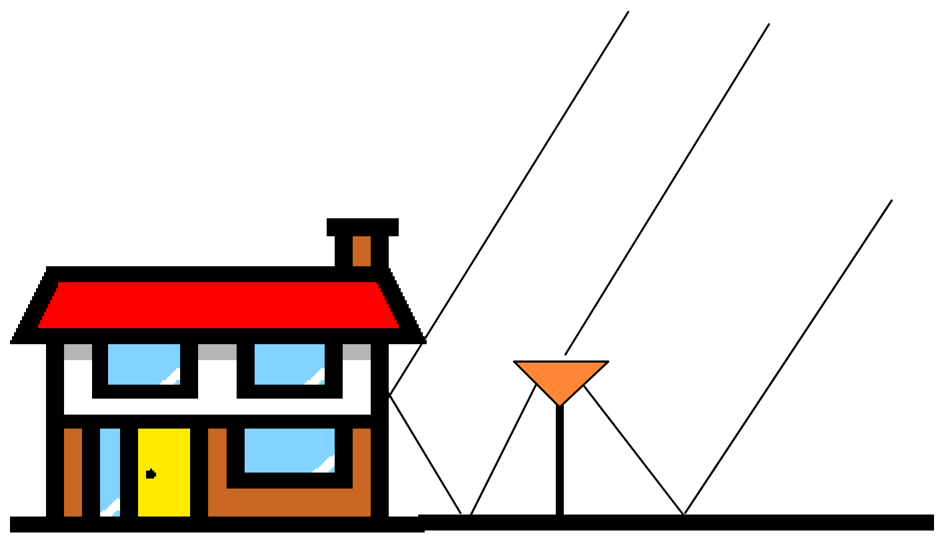
4 SVs	5 SVs	6 SVs	7 SVs	8 SVs
4.95	3.66	3.50	3.41	3.29



Position Sensors

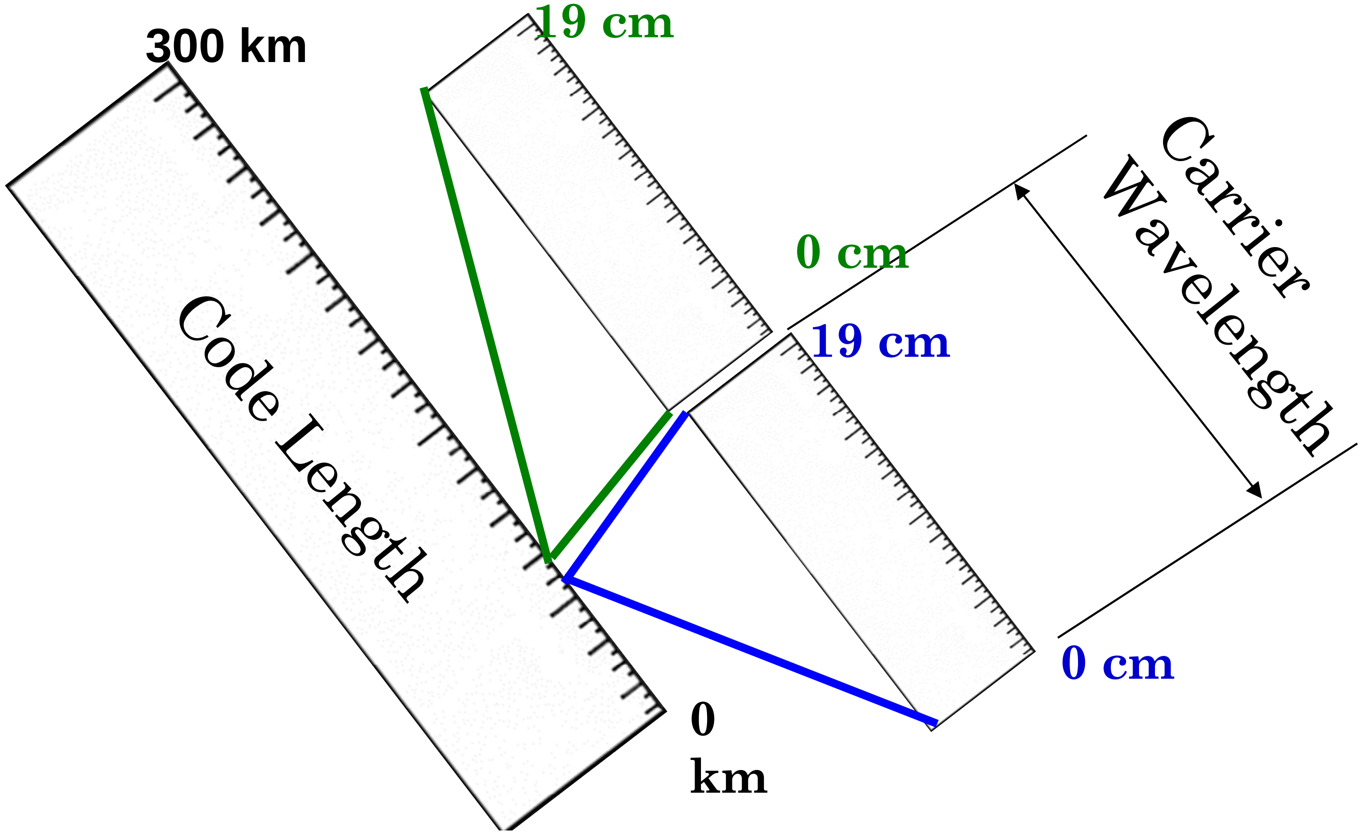
- Multi-path errors
  - Code Error: 1-5 m
    - Weakens Correlation Spike Tracking
    - Eliminated outside of 1.5 chips (500m)
  - Ground plane essential for rejection
    - Choke ring antennas used in surveying

Courtesy of Trimble Navigation



Position Sensors

- Carrier Phase Measurements
  - Its possible to track carrier phase of GPS signal along with code phase
    - e.g. Superstar II tracks to 1/1024th of a wavelength



Position Sensors

- Carrier Phase Measurements
  - Measurement Equation (wavelengths)

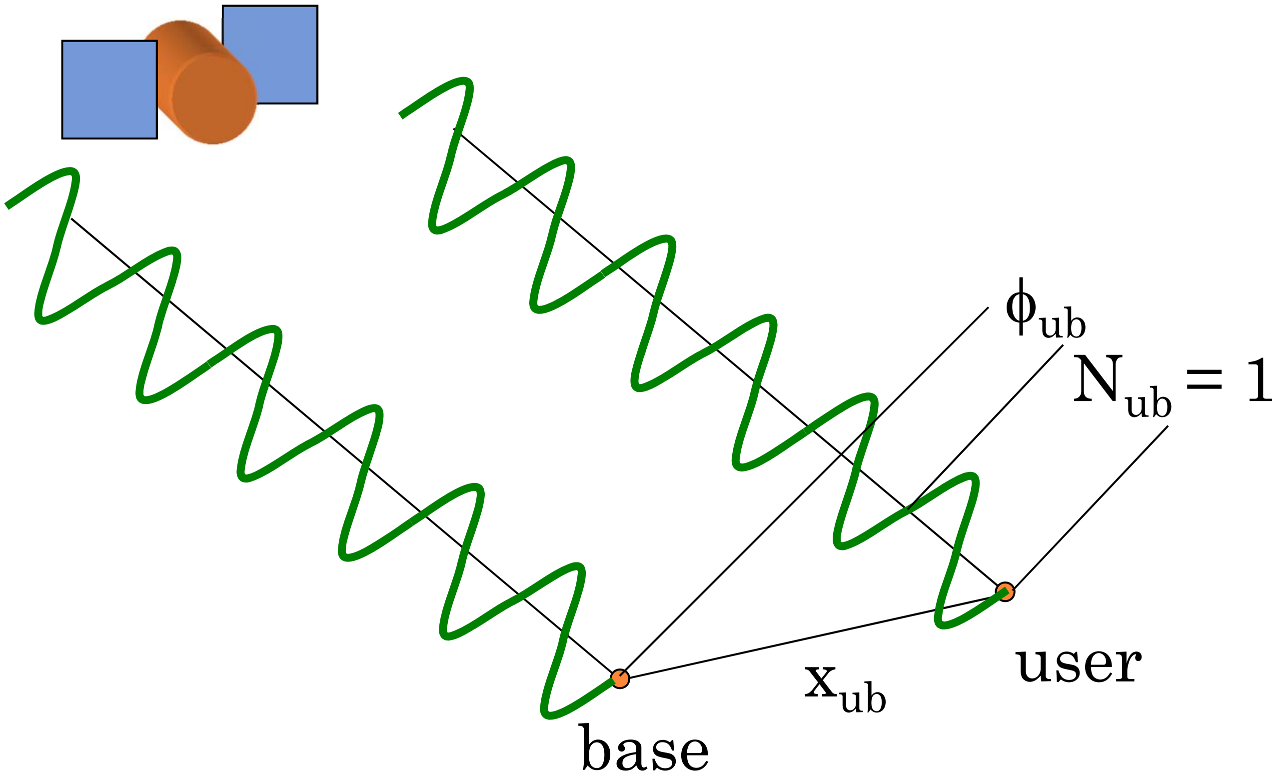
$$\phi_u^{(k)} = \lambda^{-1}\left[r_u^{(k)} - I_u^{(k)} - T_u^{(k)}\right] + f(b_u - dT^{(k)}) + N_u^{(k)} + \epsilon_{\phi,u}^{(k)}$$

Integer Ambiguity

  - Measurement precision of mm
  - Error sources in meters
  - Integers unknown, lost in errors
  - At first glance, not very useful

Position Sensors

- Single Difference Geometry



  - Single Difference removes Iono, Tropo, SV Clock, but adds base clock error
  - Double difference also removes base clock error
    - RTK (real time kinetic) systems use double difference

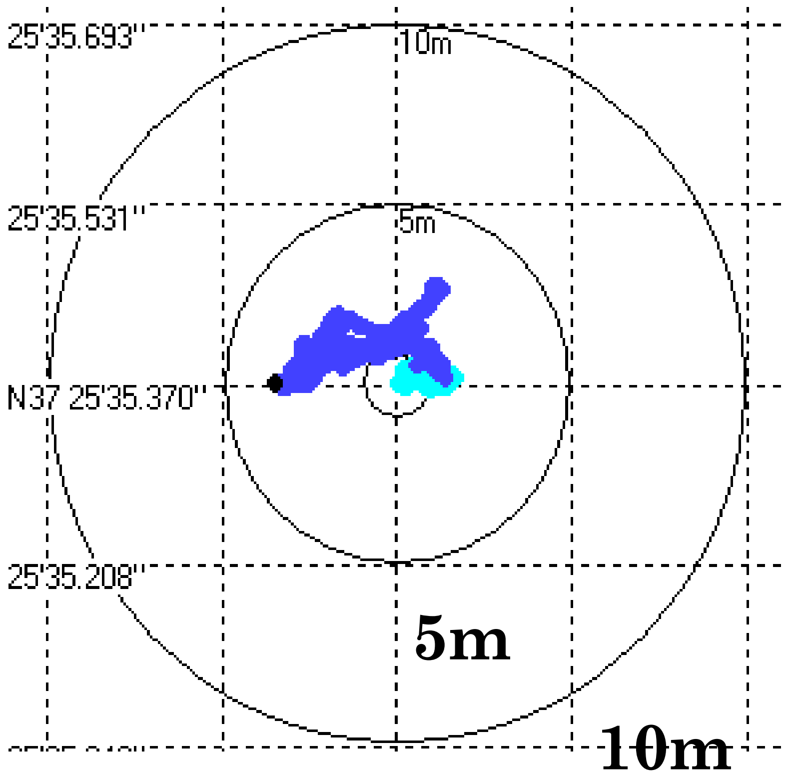
Carrier Phase Position Accuracy

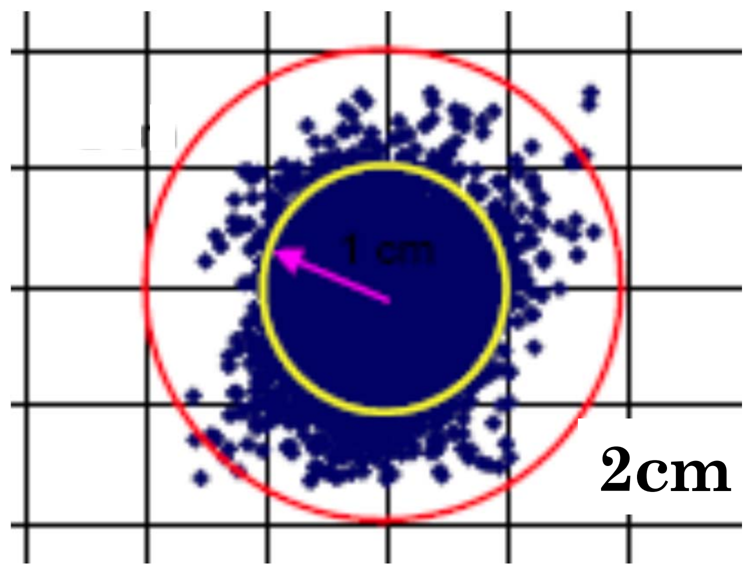
Coded Phase Solution: 25 Minutes

- Standard (3-5m CEP)
- With DGPS corrections (WAAS) (1.5m CEP)

Carrier Phase Solution: 24 Hours

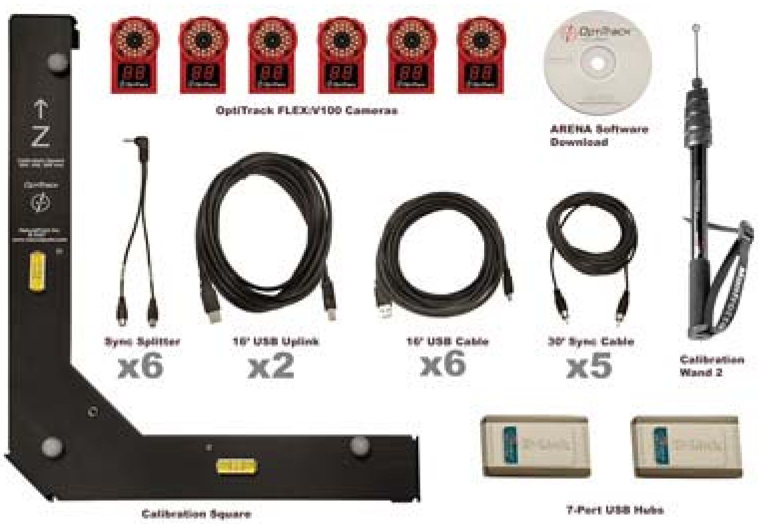
- Double Difference Technique (1cm CEP)





Position Sensors

- Indoor Positioning
  - IR Vision based Motion Capture systems
    - Vicon: returns up to 250 Hz state information for each ball, ~$50K for nominal setup
    - Optitrack: similar, but lower cost



Courtesy of Optitrack Inc.

Position Sensors

- Indoor Positioning
  - Great accuracy and precision
  - Excellent development environment, especially for indoor aerial vehicle development
    - Used for SWARM robotics at MIT (2008, How et al.)
    - Used to determine aerodynamic models of various aircraft at ETH Zurich (2007, Siegwart)
    - WAVELab, Dana Kulic, GRASP Lab, ETH, MIT, Boeing, and many more

  - Long setup time
  - Lack of flexibility
  - Limited area coverage

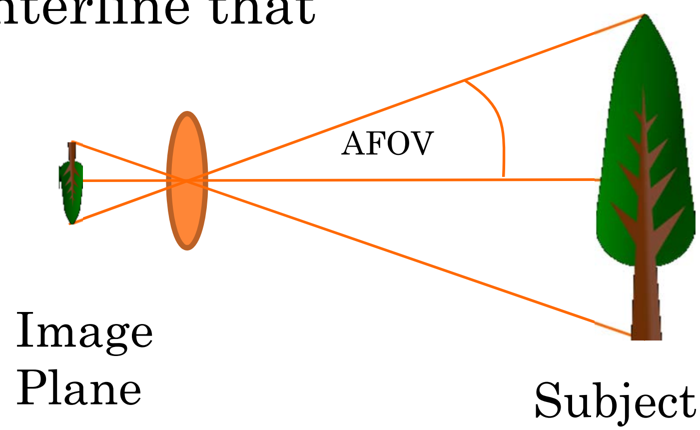
Waterloo Indoor Positioning System

Outline

- Measurement Models
  - Definition
  - Block Diagrams
  - LTI System Model
- Sensors Characteristics
- Sensor Examples
  - Contact Sensors
  - Inertial Sensors
  - Range Sensors
  - Position Sensors
  - Vision

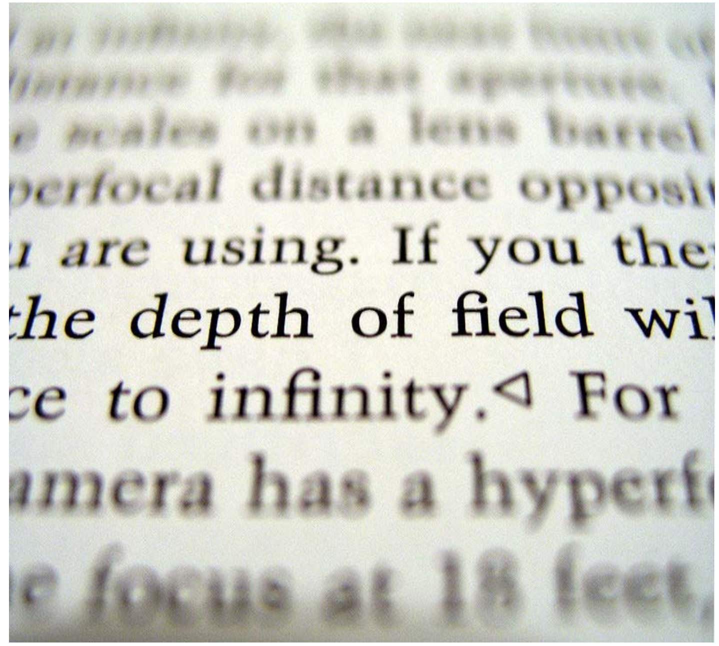
VISION BASICS

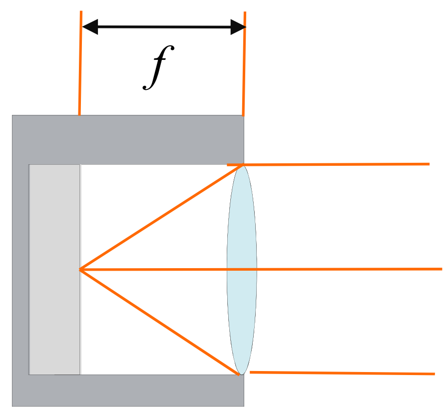
- Resolution
  - The number of pixels in the image
- Framerate
  - The number of pictures that can be taken each second
- Angular Field of View, AFOV
  - The angle from the centerline that can be imaged



VISION BASICS

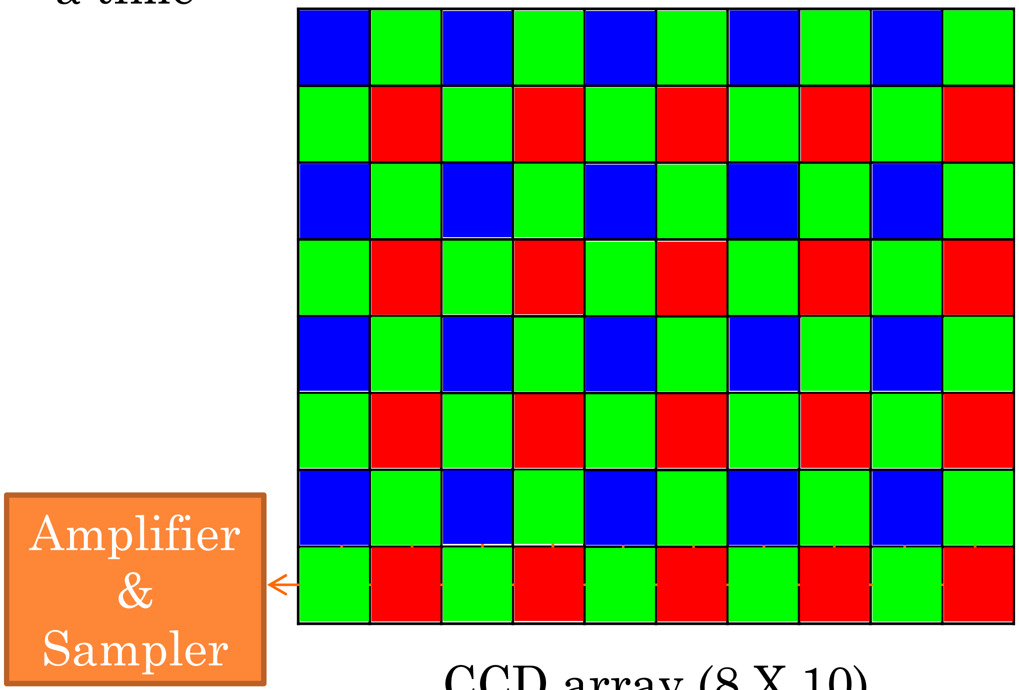
- Focus
  - Light perpendicular to the image plane is focused at the focal length
    - Source of collimated light an infinite distance from camera
  - As light sources approach, depth of focus increases
    - Image plane must be moved away from lens
- Focal Length, f
  - The distance from the lens center where parallel light is focused to a point





Vision Basics

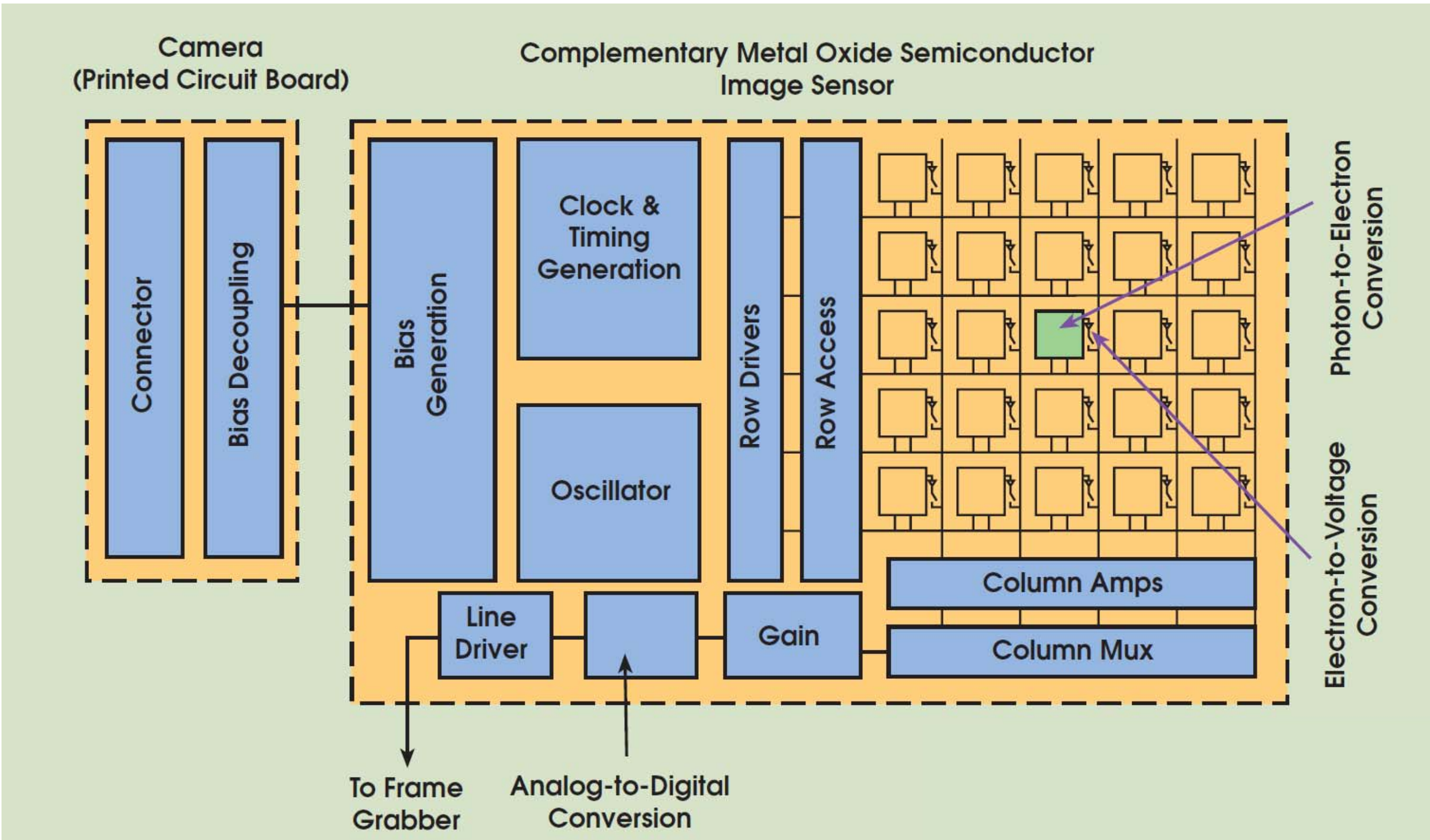
- Sensor Types
  - Charge –Coupled Device (CCD)
    - Array (or line) of capacitors which build up charge proportional to light intensity focused from a lens
    - Charges can be shifted to adjacent cells, and read off one at a time



CCD array (8 X 10)

Vision Basics

- Complementary Metal Oxide Semiconductor (CMOS)
  - Places transistors at each pixel to read them individually
  - Can access pixel voltages in any order



Vision Basics

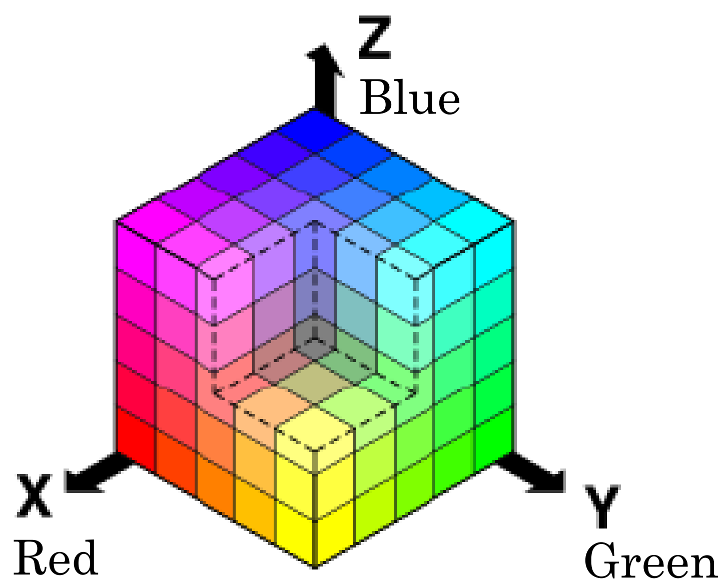
- Comparison of CCD and CMOS capabilities

Feature	CCD	CMOS
Cost	Higher – special manufacturing process	Lower – sa[illegible]s as micro[illegible]
Resolution	Higher for same sized chip	L[illegible]e sized chip
Electronic Shutter	Difficult to implement, usually requires m[illegible] shuttering	[illegible]o implement- great [illegible]r low resolution video (interlacing)
Sensitivity	Higher [illegible]ea de[illegible]absorption	Lower, up to 2/3 of chip area used by transistors
Blooming	[illegible]photons tend to [illegible]over into adjacent pixels	Inherent insensitivity to blooming due to space between pixels
Reli[illegible]ty	High, simple design of repeated element	Lower, each pixel has its own amplifier, difficult to manufacture identically
Examples	SLR Cameras, Astronomy	Webcams, Point and Shoot

Obsolete - CMOS Rules!!!

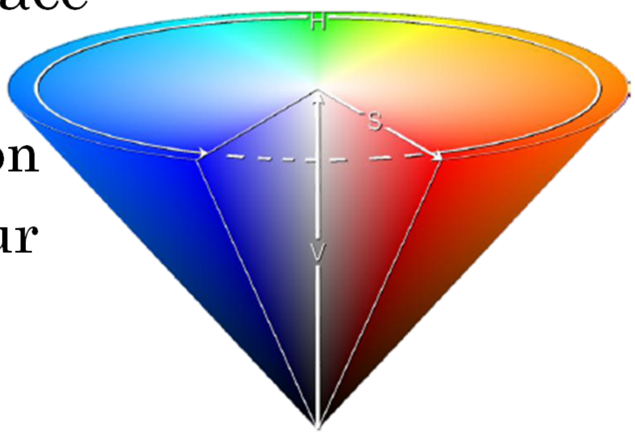
Vision Basics

- Red, Green Blue (RGB) Colourspace
  - Can define all colours in continuous space (0-100%)
  - Digital form, usually 1 byte per colour
    - 24 – bit colour yields 16 million colours
    - (0,0,0) black
    - (255, 255, 255) white
    - (255,0,0) red
    - (0,255,0) green
    - (0,0,255) blue

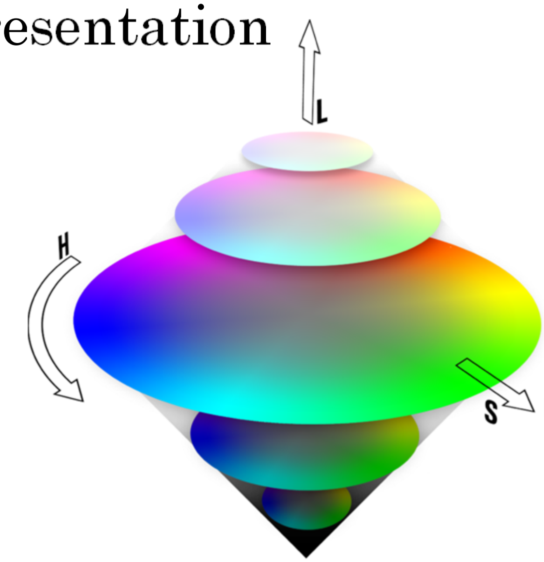


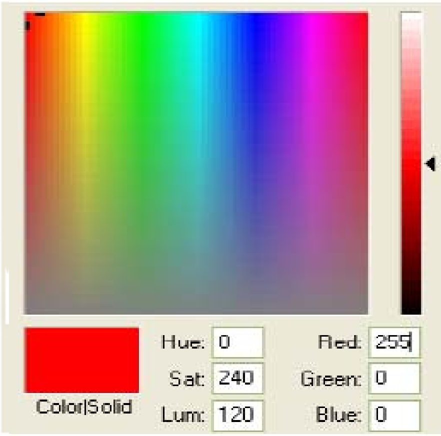
Vision Basics

- Hue, Saturation, Value Colourspace
  - Cone representation
  - Hue contains chromatic information
  - Saturation defines "purity" of colour
  - Value defines "strength" of colour



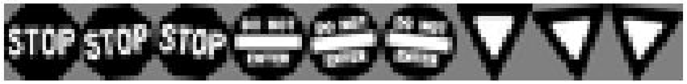
- Hue Saturation, Luminance, Colourspace
  - Double cone representation



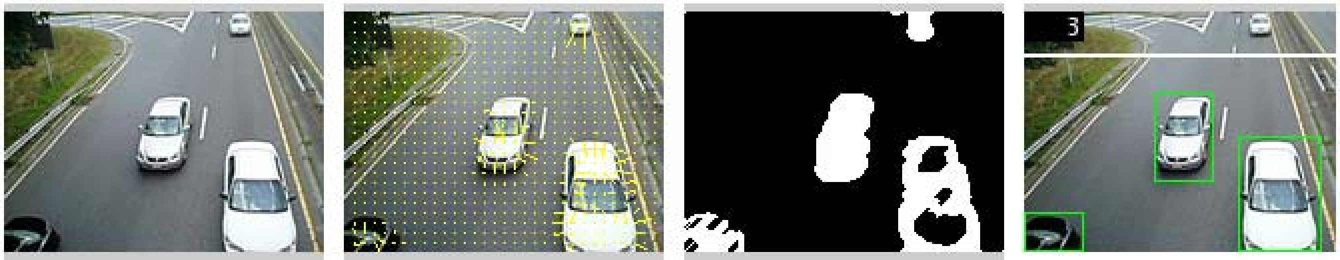


Vision Basics

- Tools
  - Matlab Image Processing Toolbox & Simulink Video and Image Processing Blockset
    - A great way to learn, test, practice
    - Very easy to generate graphics, manipulate data, analyze individual images
    - Many interesting demos
      - Lane Departure Warning
      - Traffic Sign Detection

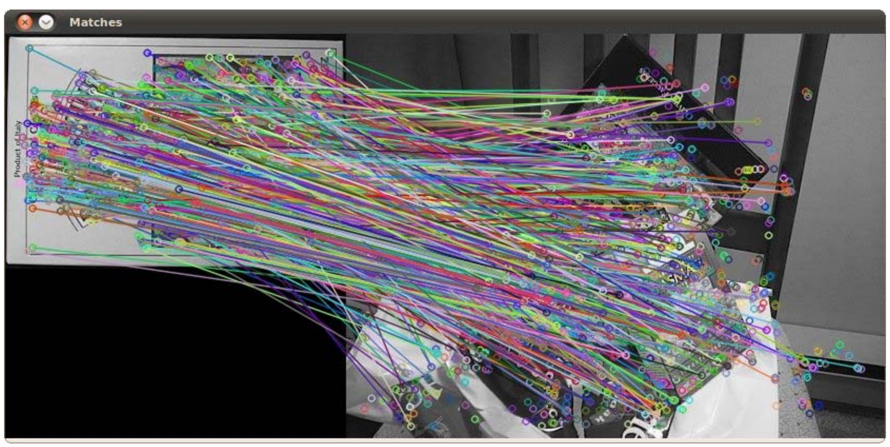


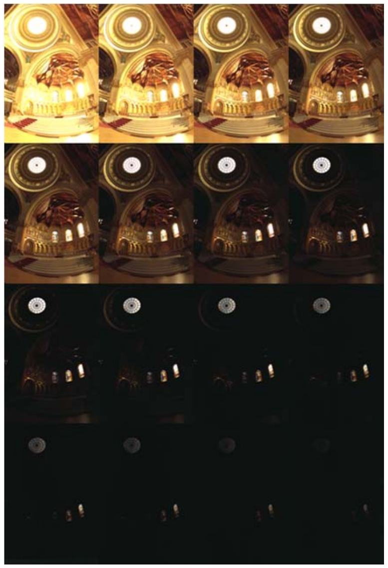
      - Tracking cars using optical flow



Vision Basics

- Tools
  - OpenCV (Linux or Windows): image processing in C++
    - Handles much of the low-level data conversions
    - Provides a solid library of advanced features
    - HighGUI: Very handy GUI for debugging, displaying efforts
    - Not necessarily optimized for performance, but getting better
    - Automatic parallelization through CUDA
    - ROS integrated

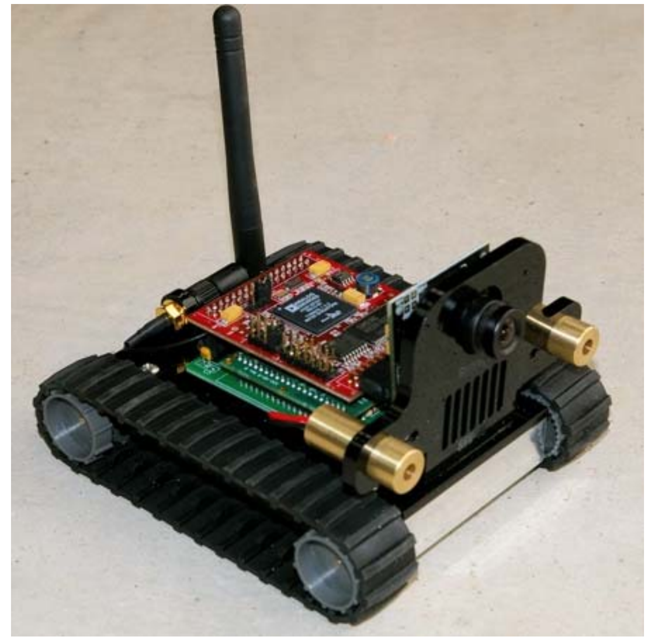




Vision

- One image contains an amazing amount of environment information
- Image sequence: video
  - Environment information and clues about vehicle motion
- Vision on mobile robots used for
  - Tracking
  - Mapping
  - Obstacle detection
  - State estimation
- Computer vision is an enormous field with thousands of researchers, hundreds of useful techniques
- Many strong vision researchers here at Waterloo

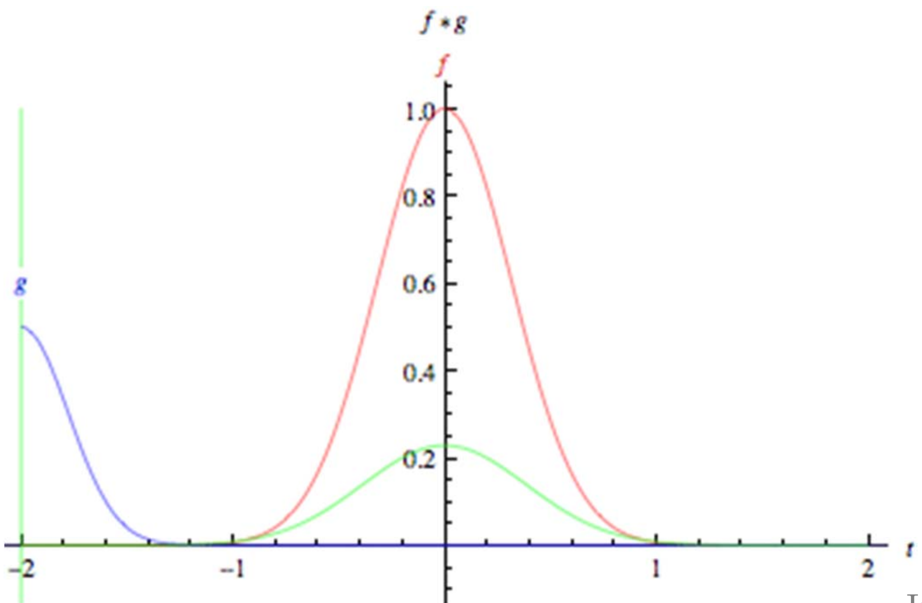
Surveryor SRV-1 Blackfin



VISION

- Convolution
  - A fundamental operation for many vision processing methods
  - A discrete implementation of the continuous convolution concept

$$[f * g] = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$$



Convolution (green) of two Gaussians (red and blue). Grey line is integrand, area represents integral at *t*.

Image courtesy of Wolfram Research

VISION

- Discrete Convolution
  - Applied to images $$(f*g)[n]=\sum_{m=-\infty}^{\infty} f[m]\cdot g[n-m]$$

$$(I*K)[i,j]=\sum_{k=-p}^{p}\sum_{l=-q}^{q} I(i+k,j+l)K(k,l)$$

K_{11}	K_{12}	K_{13}	I_{14}	I_{15}	I_{16}	I_{17}	I_{18}	I_{19}
K_{21}	K_{22}	K_{23}	I_{24}	I_{25}	I_{26}	I_{27}	I_{28}	I_{29}
K_{31}	K_{32}	K_{33}	I_{34}	I_{35}	I_{36}	I_{37}	I_{38}	I_{39}
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
I_{k1}	I_{k2}	I_{k3}	I_{k4}	I_{k5}	I_{k6}	I_{k7}	I_{k8}	I_{k9}

Image (Intensities)

K_{11}	K_{12}	K_{13}
K_{21}	K_{22}	K_{23}
K_{31}	K_{32}	K_{33}

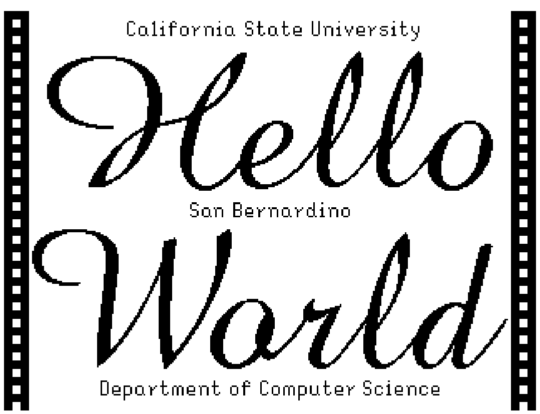
Convolution Kernel

VISION

- Convolution Example
  - Gaussian Blur
  - Gaussian Kernel (7x7, $\sigma=0.8408$)

$K_G=$

0.00000067	0.00002292	0.00019117	0.00038771	0.00019117	0.00002292	0.00000067
0.00002292	0.00078633	0.00655965	0.01330373	0.00655965	0.00078633	0.00002292
0.00019117	0.00655965	0.05472157	0.11098164	0.05472157	0.00655965	0.00019117
0.00038771	0.01330373	0.11098164	0.22508352	0.11098164	0.01330373	0.00038771
0.00019117	0.00655965	0.05472157	0.11098164	0.05472157	0.00655965	0.00019117
0.00002292	0.00078633	0.00655965	0.01330373	0.00655965	0.00078633	0.00002292
0.00000067	0.00002292	0.00019117	0.00038771	0.00019117	0.00002292	0.00000067



Gaussian blur with $\sigma=2$.
Images from Wikipedia

VISION

- Three of the main classes of vision algorithms for motion estimation
  - Monocular vision
    - Track features from image to image as vehicle moves
    - Known/unknown vehicle motion
    - Major issue is correspondence
    - Scale ambiguity
  - Stereo Vision
    - Using two cameras with fixed baseline, calculate position of common features/every pixel
    - Correspondence issue is reduced (only match along horizontal lines
  - Optical Flow
    - Track motion of brightness patterns in image
    - Can estimate vehicle motion and proximity to obstacles
  - Many others for scene understanding: clustering, segmentation, classification, labeling, reconstruction

VISION

- Feature Identification
  - Many feature extraction options exist, most popular
    - Corners: **Harris**, Shi-Tomasi, ZLoG, DoG
    - Edges: Canny, **Sobel**
    - Others: **SIFT**, Speeded Up Robust Features (SURF, U-SURF, SURF-128), FAST, STAR, MSER, GFTT
      - Significantly more computation than other features, but provide possibility of unique identification and persistence

VISION

- Feature Identification
  - Harris Corners
    - Test each pixel for similarity to surrounding pixel areas
    - Considers the sum of square differences

$$S[i,j] = \sum_{k=-p}^{p} \sum_{l=-q}^{q} (I[k+i, j+l] - I[i,j])^2$$

    - Strongest S score defines most likely corners (most trackable features, not necessarily a corner)

Vision

- Harris Corners

VISION

- Feature Identification
  - Sobel Edge Detection: gradient of the image
  - Convolution of Image with two kernels
    - Horizontal and Vertical Derivative approximations

$$E_x = I * K_x, E_y = I * K_y$$

$$K_x = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix}, K_y = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

Vision

- Feature Identification
  - Sobel Edges
  - Magnitude and direction of the gradient can be used
    - Magnitude gives most reliable indication of edges

$$E(i,j) = \sqrt{E_x^2(i,j) + E_y^2(i,j)}$$

    - Direction can be used to define lines

$$\theta[i,j] = \tan^{-1}\left(\frac{E_y[i,j]}{E_x[i,j]}\right)$$

VISION

- Sobel Edge Detector

Vision

- More Advanced Feature Identification
  - Scale Invariant Feature Tracking (SIFT)
    - Multiple Gaussian Blurs performed at different scales
    - Difference of Gaussians computed (DoG)
    - Maxima and Minima are scale invariant features
    - Filter out low contrast points
    - Eliminate edge points (often not persistent)

DoG Maxima & Minima

Remove Low Contrast

Remove Edge points

Vision

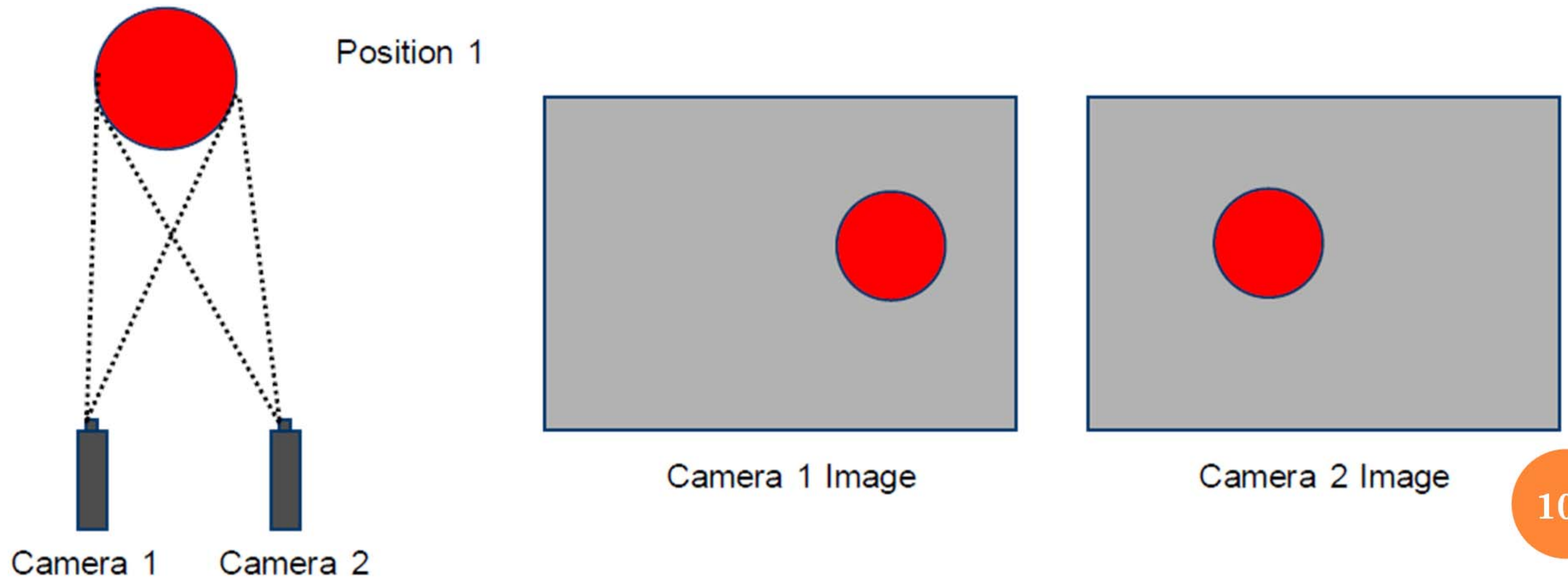
- Monocular vision
  - Feature tracking on GPUs (courtesy of EPI Lab)

Harris corners on GPU

SIFT, FAST, BRIEF, ORB Features

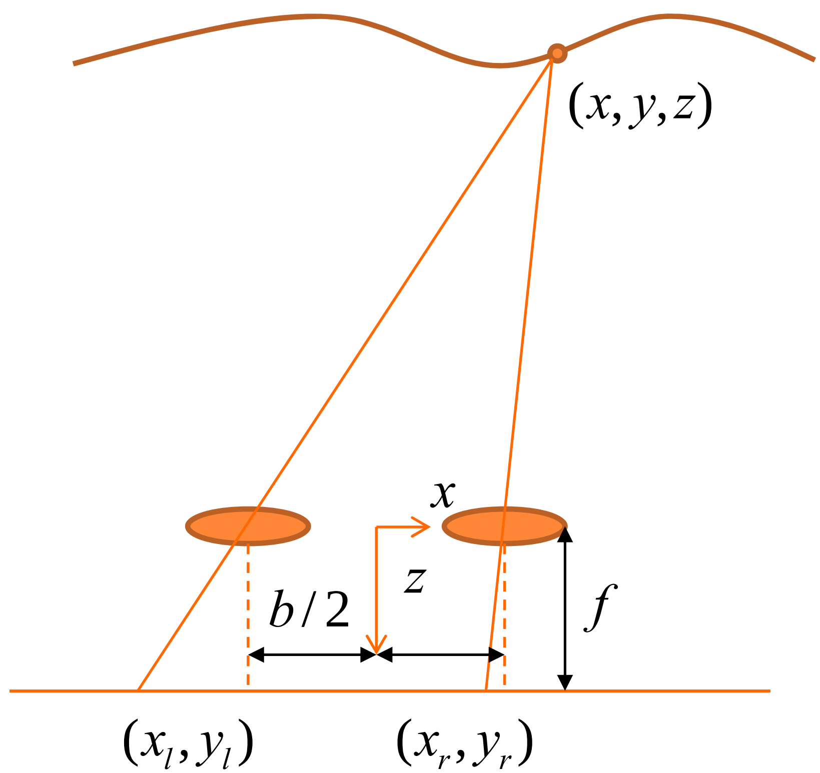
VISION

- Stereo Vision
  - Addresses scale issue, must match features/pixels between images and triangulate
  - Better at close range (humans only use stereo up to an arms length away)
  - Images of a red ball with two cameras



Vision

- Stereo Vision
  - Consider idealized cameras
  - Compare projection of point on two image planes



Vision

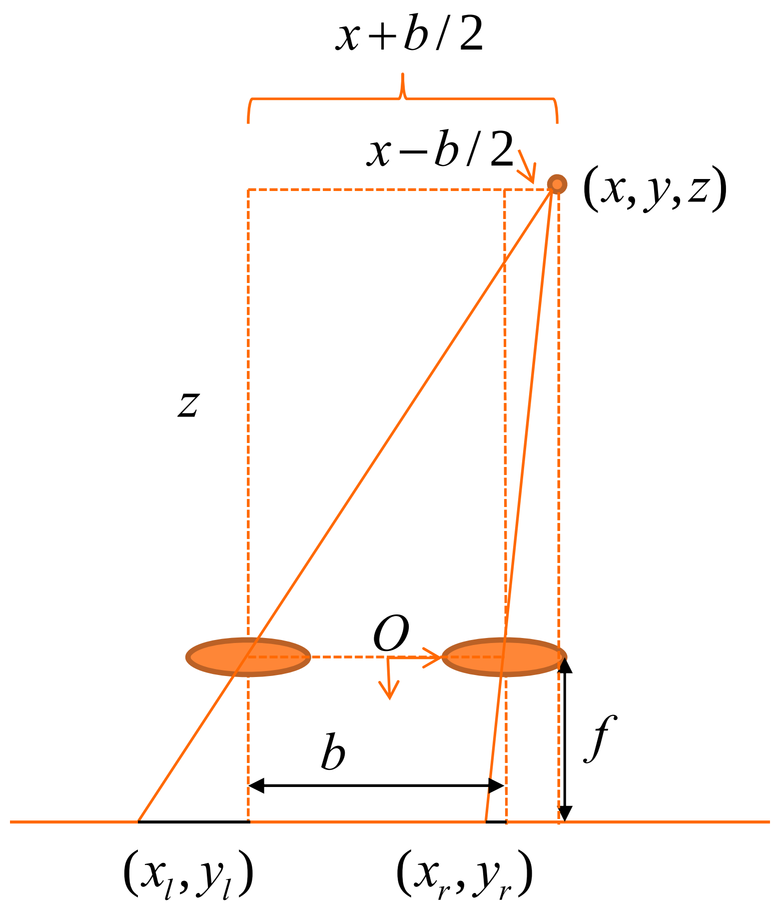
- Stereo Vision
  - In xz plane

$$\frac{x_l}{f} = \frac{x + {}^{b}\!/_{2}}{z}, \frac{x_r}{f} = \frac{x - {}^{b}\!/_{2}}{z}$$

  - Combining yields

$$\frac{x_l - x_r}{f} = \frac{b}{z}$$

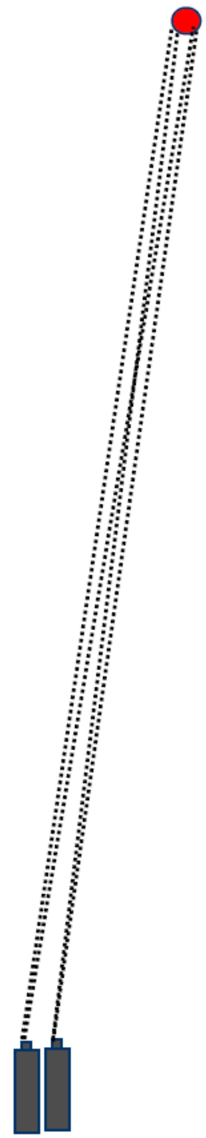
  - Solving for x,y,z



$$x = \frac{b(x_l + x_r)}{2(x_l - x_r)}, \quad y = \frac{b(y_l + y_r)}{2(x_l - x_r)}, \quad z = \frac{bf}{(x_l - x_r)}$$

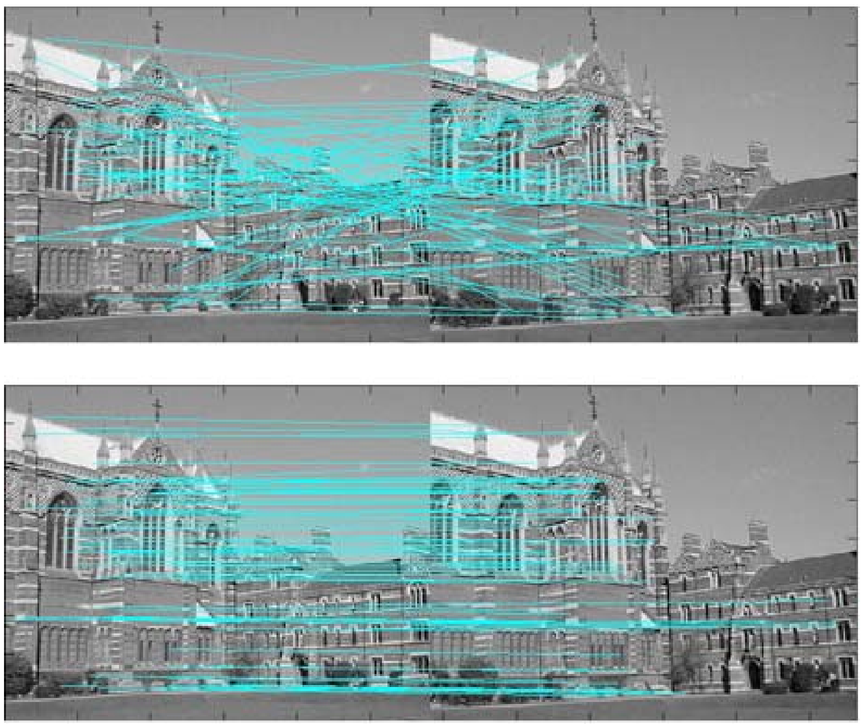
VISION

- Stereo Vision
  - Disparity: difference between x_l and x_r
  - The farther away an object, the smaller the disparity
  - Poor quality depth information when baseline is small compared to depth
  - Similar to monocular vision at long range (> 15 times baseline)
  - Disparity is proportional to b
    - For longer range need larger baseline
    - This is why humans only use stereo for tasks at the end of their hands!



VISION

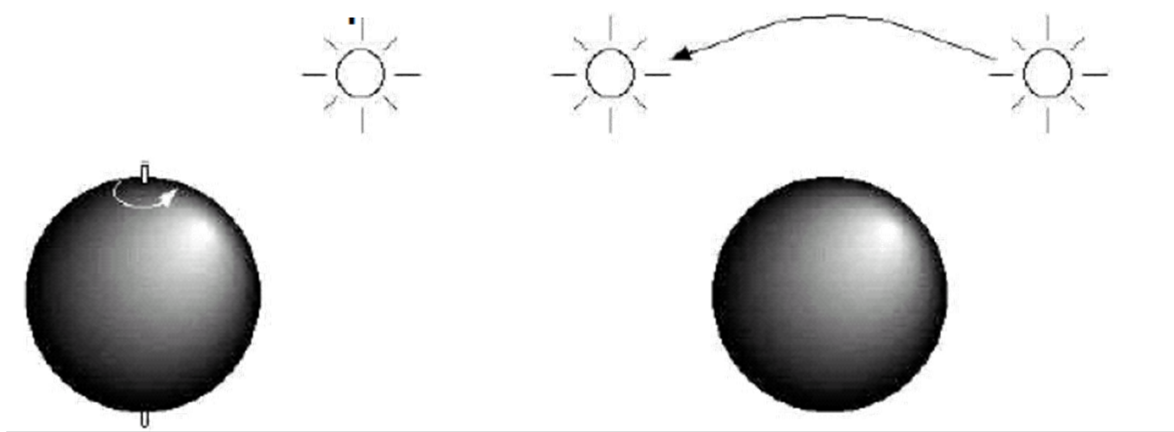
- Stereo Vision
  - Feature Association
    - Must be able to identify the same features in two separate images
    - SIFT matches, best match vs. restricted baseline



Courtesy: Dr. Carl Salvaggio, Rochester Institute of Technology

Vision

- Optical Flow
  - Motion of Brightness patterns in image
  - Can be caused by environment or light source motion
- Motion Field
  - A field of velocity vectors that defines the motion of each point in the environment relative to the camera frame.
  - Velocity of the environment can be related to velocity in the image plane.



VISION

- Optical Flow
  - Given two consecutive images $I(t), I(t+dt)$
  - Find u,v such that for some subset of the image

$$I(x+udt, y+vdt, t+dt) = I(x,y,t)$$

    - over a constant intensity patch

  - Using Taylor series expansion, we get

$$I(x,y,t) + dx\frac{\partial I}{\partial x} + dy\frac{\partial I}{\partial y} + dt\frac{\partial I}{\partial t} + H.O.T. = I(x,y,t)$$

  - Divide through by *dt*. In the limit, as t tends to zero

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

VISION

- Optical Flow
  - Abbreviating: $$\frac{\partial I}{\partial x}\frac{dx}{dt}+\frac{\partial I}{\partial y}\frac{dy}{dt}+\frac{\partial I}{\partial t}=0$$

$$I_x u + I_y v + I_t = 0$$

  - Known as the optical flow constraint equation
    - Need to find u,v for each point based on I_x, I_y, I_t
    - Only one constraint for two variables, need another assumption

VISION

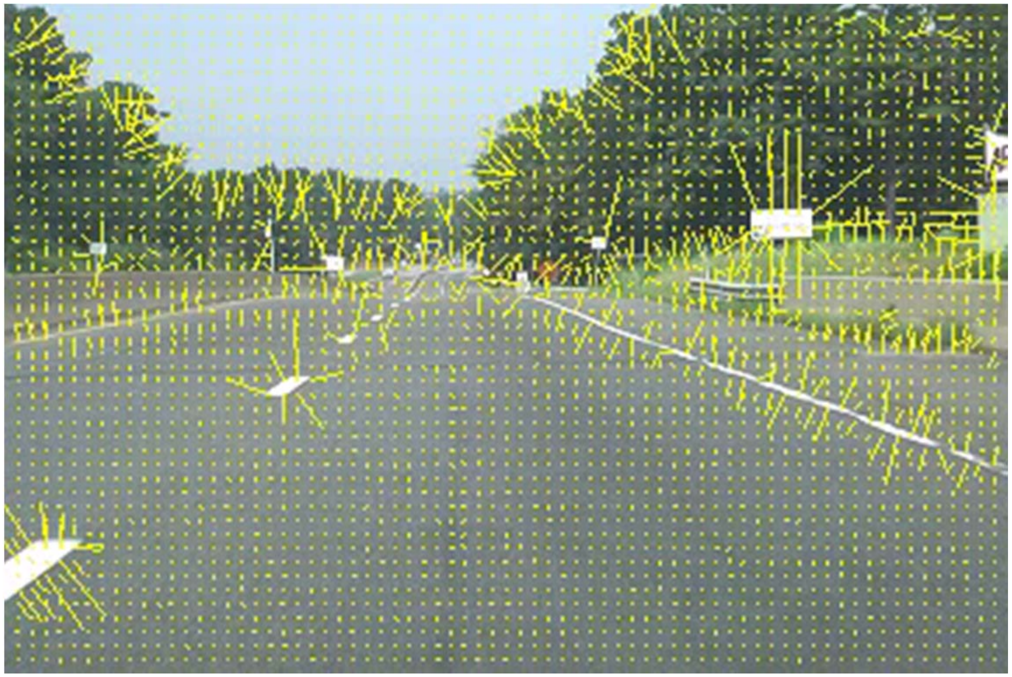
- Optical Flow – Lucas-Kanade (1981)
  - Assume motion is the same for a small window around each pixel (*mXm*, $n=m^2$)

$$\begin{bmatrix} I_{x,1} & I_{y,1} \\ I_{x,2} & I_{y,2} \\ \vdots & \vdots \\ I_{x,n} & I_{y,n} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -I_{t,1} \\ -I_{t,2} \\ \vdots \\ -I_{t,n} \end{bmatrix}$$

  - Can solve for u,v using least squares
  - Can weight central pixels more than outer edge.

Vision

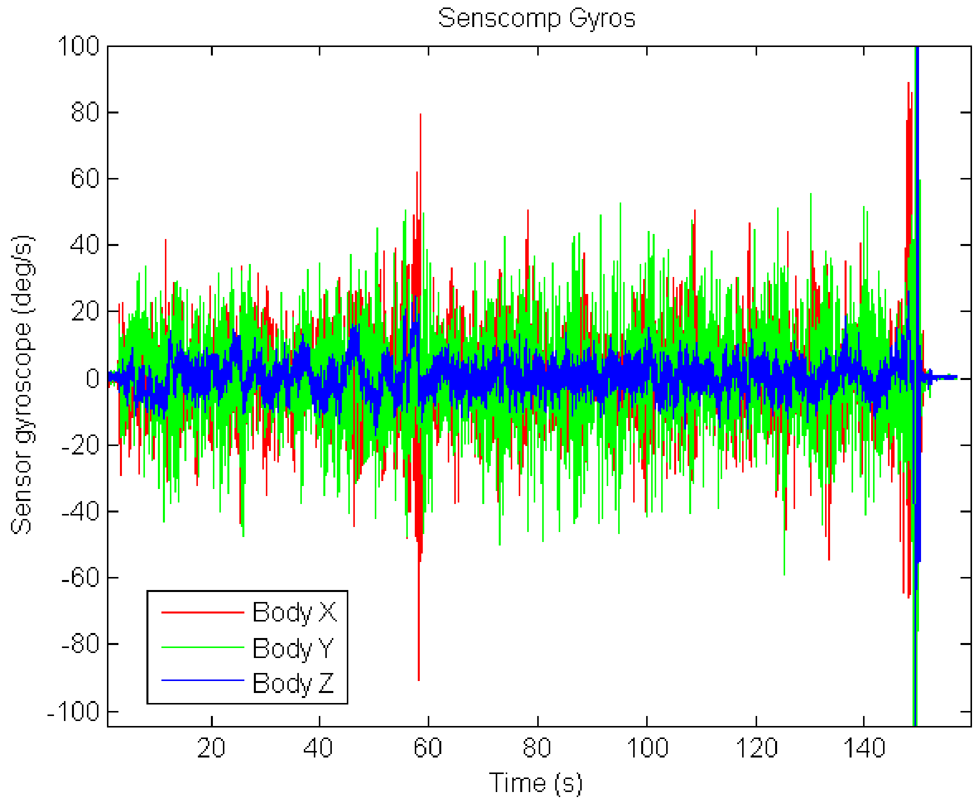
- Optical Flow
  - Key drawback is the implicit assumption that all objects are moving at the same speed
  - Also a local method that depends on second derivative information, can be noisy, doesn't work inside uniform patches.
  - Can be used to estimate vehicle motion



Extra Slides

INERTIAL SENSORS

- Gyroscope Data
  - Aeryon Scout quadrotor in hover flight
    - Vibration affects x, y axes more than z axis
    - Angular rate motion clearly visible (good signal!)



Position Sensors

- Wheel Odometry
  - Captures rotation as a series of digital pulses
  - Consists of a rotating clear disk with opaque bands, a light source, and a photo sensor
  - Light shining through the transparent band triggers the photo sensor, which outputs a digital pulse
  - Two types of rotary encoders:
    - Incremental (shown at right)
    - Absolute

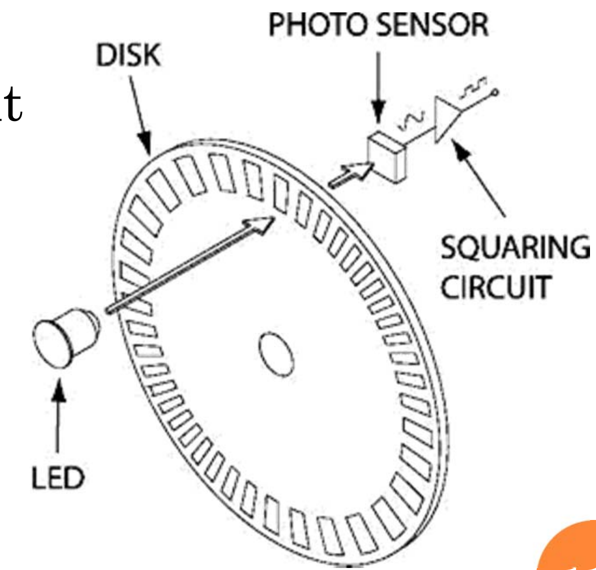
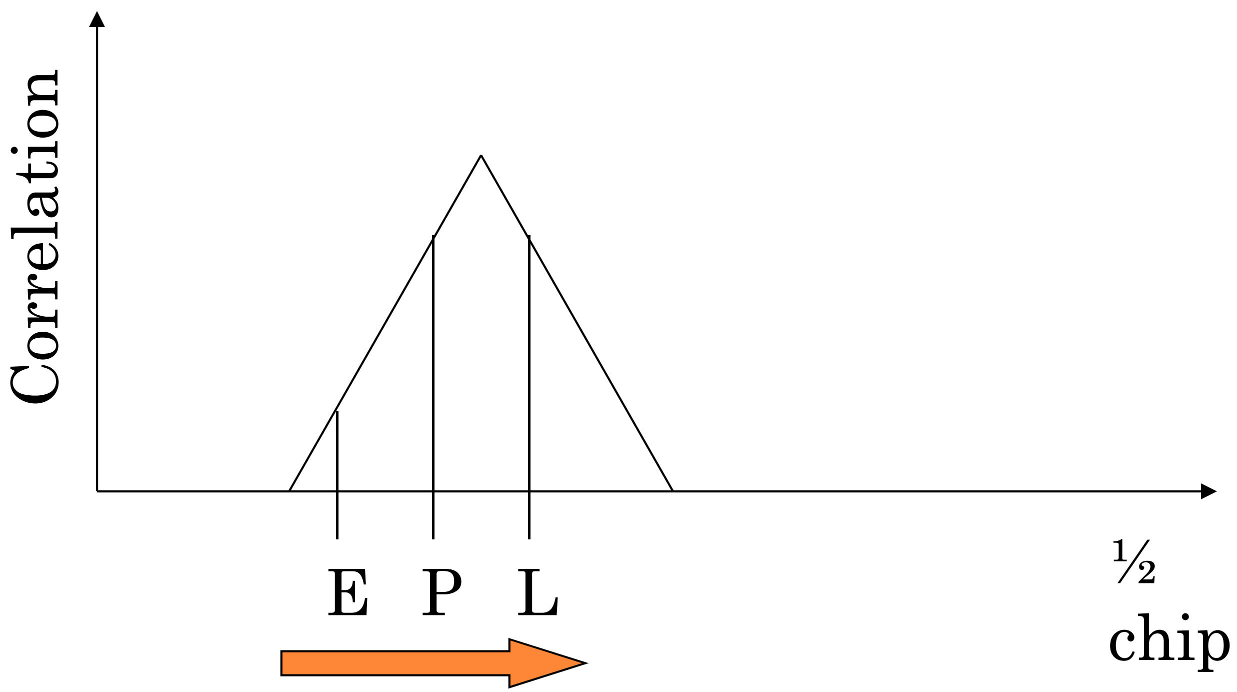


Image courtesy of Encoder Products Company

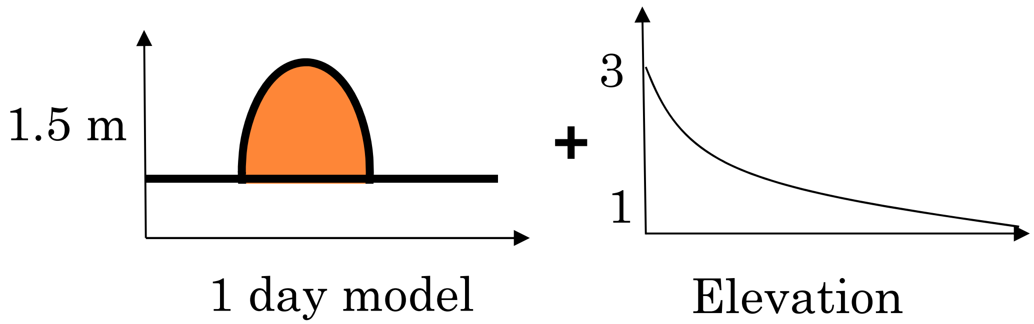
Position Sensors

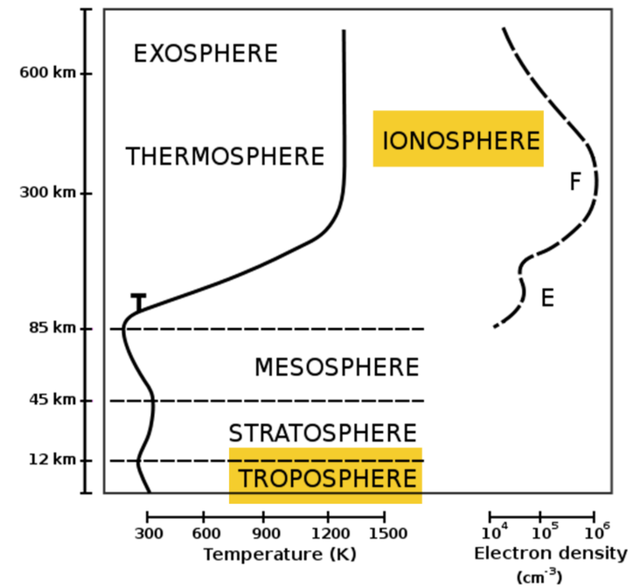
- Autocorrelation Peak Tracking
  - Early, Prompt, Late by ½ chip
  - Resolution on tracking loop allows for measurement of 1/1024th of a ½ chip



Position Sensors

- Ionosphere (1.5-10 m)
  - Dispersive delay
  - Klobuchar Daytime Model and
  - Average Obliquity Factor





- Troposphere (1-4 m)
  - Dry and Wet Air Delays

- Differential corrections can effectively eliminate atmospheric errors
  - Ground beacons, base stations
  - Satellite Based Augmentation Systems (SBAS): WAAS, OmniStar

Position Sensors

- Satellite Position Calculation
  - 15 Ephemeris Data Elements
    - Time of ephemeris
    - 5 orbital parameters
    - 9 correction terms
  - 1-2 m position accuracy along measurement
  - Angles specified in semicircles, so that π can be used consistently in calculations

$$\pi = 3.1415926535898$$

  - SV Position at Time of Transmission

$$T_{tr} = T_u - \frac{\rho}{c}$$

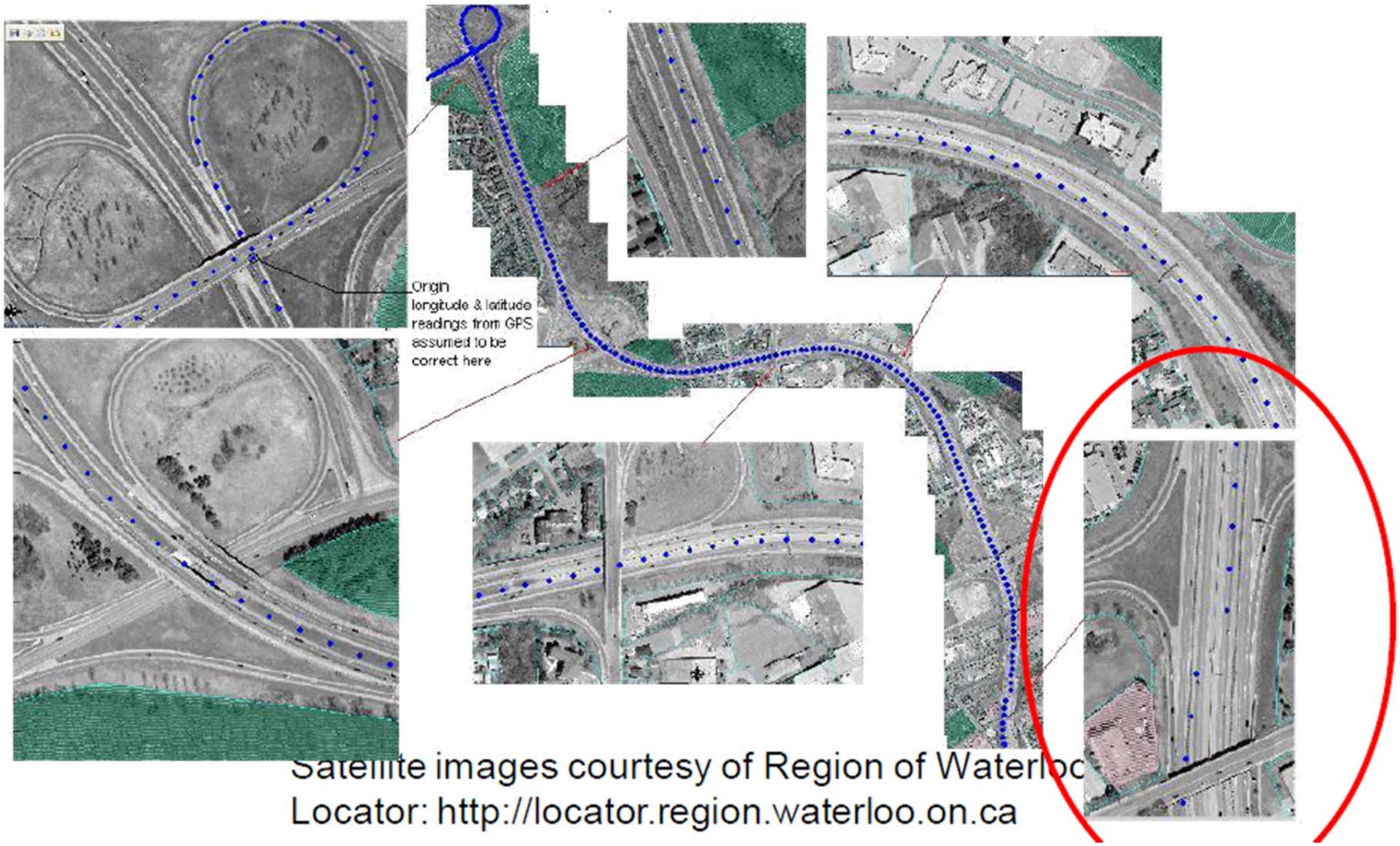
Position Sensors

- Satellite Clock Corrections
  - Time of Clock Correction
  - Quadratic correction coefficients
  - Relativistic Effect
    - Due to orbit eccentricity (variation in SV speed)

$$
\begin{aligned}
dT &= af_0 + af_1(T - T_{tr}) + af_2(T - T_{tr})^2 \\
&\quad + \Delta T_r
\end{aligned}
$$

POSITION SENSORS

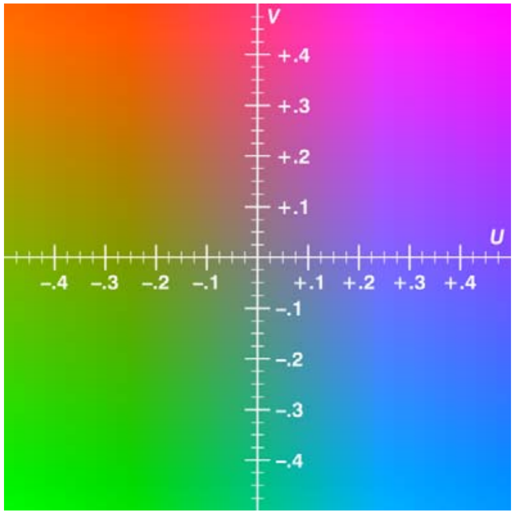
- GPS Data – no DGPS
  - Driving around town (5-10m 50% CEP)



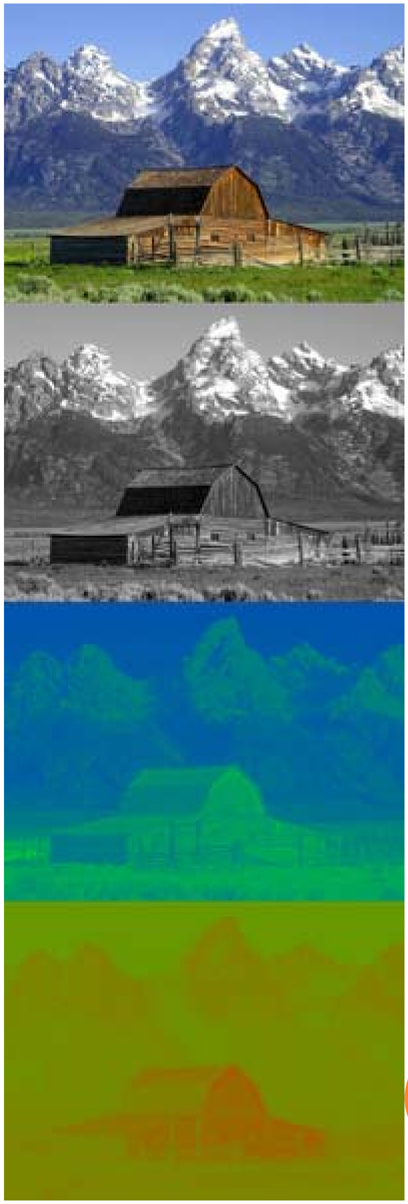
Satellite images courtesy of Region of Waterloo
Locator: http://locator.region.waterloo.on.ca

Vision Basics

- YUV, YCbCr Colourspaces
  - Human focused colourspace, invented during switch to colour TV
    - Humans are more sensitive to brightness than to colour, can downsample colour channels
  - Some webcams output raw YCbCr
    - Logitech Quickcam Pro 9000



UV map at Y = 0.5



VISION

- Monocular Vision
  - Tracking moving objects
    - Colour Blob Tracking
      - Difficult in RGB, better in HSV
        1. Select desired hue
        2. Select desired hue window
        3. Search for all pixels in hue window
        4. Reject pixels with extreme saturation

Vision

- Cone Detection

VISION

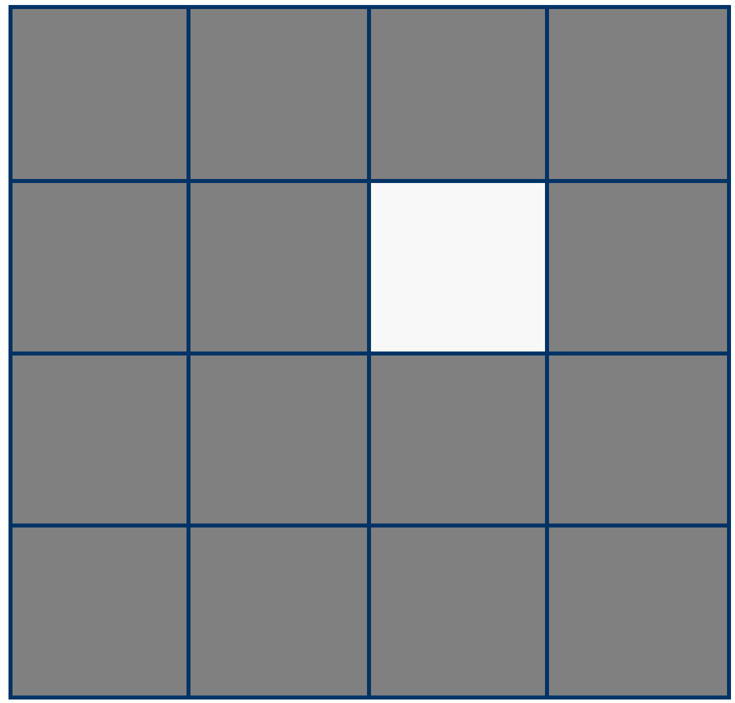
- Monocular Vision
  - Target tracking with static cameras
    - Can be used for indoor positioning systems
      1. Background subtraction using absolute difference of RGB values
      2. Convert image to binary via thresholding
         1. Static Threshold
         2. Otsu's Method (minimum intraclass variance)
      3. Find boundaries of connected pixels (brute force search)

Vision

- Automobile Tracking

VISION

- Feature Identification
  - Zero Crossing of Laplacian of Image Gaussian (ZLoG)
  - Consider the following greyscale image
  - Identify the bright spot using ZLoG

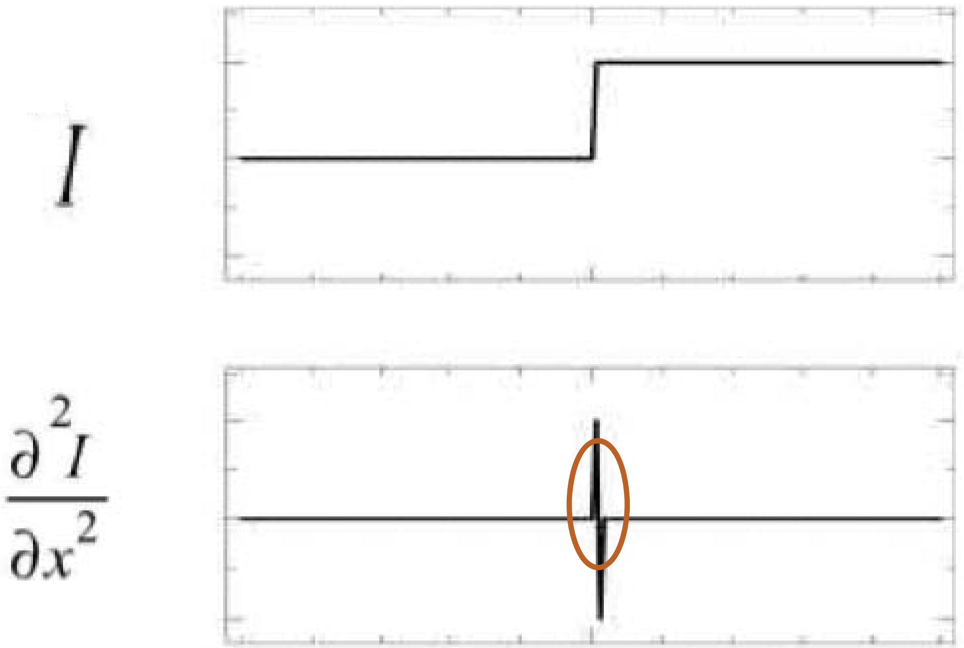


$$I(i,j)=\begin{cases}0.8 & i=j=3\\ 0.1 & otherwise\end{cases}$$

VISION

- Feature Identification
  - Laplacian:
    - Divergence of gradient

$$L(x,y) = \nabla^2 I(x,y) = \frac{d^2 I}{dx^2} + \frac{d^2 I}{dy^2}$$



  - The zero crossing highlights a significant change

VISION

- Feature Identification
  - Laplacian kernel approximation:

K_L=

0	1	0
1	-4	1
0	1	0

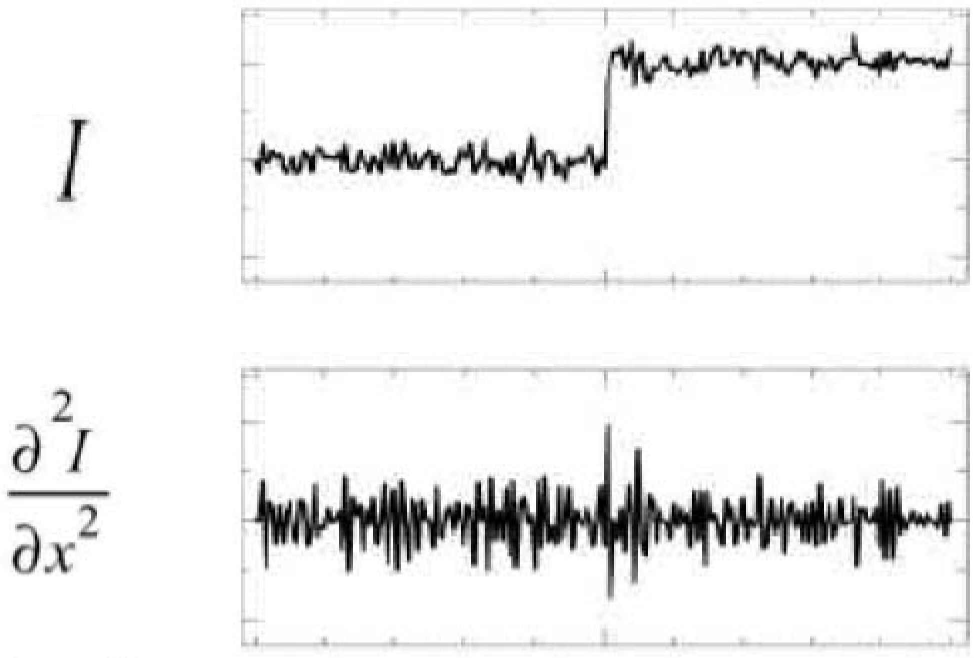
- Applied to our example image

0.0	0.0	0.7	0.0
0.0	0.7	-2.8	0.7
0.0	0.0	0.7	0.0
0.0	0.0	0.0	0.0

Zero crossing

VISION

- Feature Identification
  - In real images, noise is a significant problem for zero crossing detection
    - Laplacian uses second derivative information
      - This amplifies pixel noise signficantly

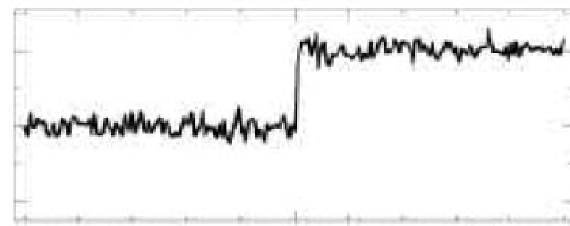


VISION

- Feature Identification
  - To deal with noise, prefilter image with Gaussian blur

$$K_G = \begin{bmatrix} \frac{1}{16} & \frac{2}{16} & \frac{1}{16} \\ \frac{2}{16} & \frac{4}{16} & \frac{2}{16} \\ \frac{1}{16} & \frac{2}{16} & \frac{1}{16} \end{bmatrix}$$

I



$I * K_G$

Vision

- Optical Flow – Horn-Schunk (1980)
  - Additional assumption: Smoothness
    - Essentially assuming that all objects in the image move in the same way
    - Violated when there are objects moving in different directions
    - Violated when objects are at different depths when camera moves
  - Define smoothness error: integral of square of magnitude of gradient of optical flow

$$e_s = \iint \left(u^2 + v^2\right) dxdy$$

  - Define constraint error: integral of constraint error squared

$$e_c = \iint \left(I_x u + I_y v + I_t\right)^2 dxdy$$

VISION

- Optical Flow
  - To satisfy these two requirements as much as possible, minimize:

$$e_s + \lambda e_c$$

  - Lambda is a weighting parameter that is adjusted manually (large if intensity is accurate)
  - Calculus of variations yields a pair of elliptical second-order PDEs, which can be solved iteratively.

$$\nabla^2 u = \lambda(I_x u + I_y v + I_t) I_x$$

$$\nabla^2 v = \lambda(I_x u + I_y v + I_t) I_y$$

- Recall, Laplacian operator: $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

Vision

- Optical Flow
  - Discrete Implementation Algorithm of Horn-Schunk
    - Calculate x, y derivatives through convolution
    - Apply Gauss-Siedel iterative solver to determine flow field

$$u^{k+1} = \frac{\Delta u^k - \frac{1}{\alpha} I_x \left(I_y v^k + I_t \right)}{\frac{1}{\alpha} I_x^2}$$

$$v^{k+1} = \frac{\Delta v^k - \frac{1}{\alpha} I_y \left(I_x u^k + I_t \right)}{\frac{1}{\alpha} I_y^2}$$

- A simple way to find the change in the flow field is to look at a neighbourhood of the four adjacent pixels

$$\Delta u(i,j) = \frac{1}{4} \sum_{N(i,j)} u(N(i,j)) - u(i,j)$$